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Some Topological Properties of Fuzzy Anti Normed Linear Spaces

بعض الخصائص التبولوجية للفضاءات الخطية غير المنتظمة الغائمة

A thesis Submitted in Partial Fulfillment of the Requirements for the Degree of M. Sc. In Mathematics

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Dedication

To my family, my husband and chiding

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I would also like to express my special thanks of gratitude to my supervisor Dr. **Adel Ahmed Hassan Kabba** for the true effort in supervising and directing me to come with this thesis. Also, my great thanks are due to my family for their support, encouragement and great efforts for me. I would like also to express my gratitude to all those who supported and helped me in completing this study.

Abstract

We study fuzzy Topological spaces. We show some topological and Algebraic of a fuzzy subset topology. We obtain some topological properties of fuzzy anti normed linear spaces. We mention some applications of topological properties of fuzzy anti normed linear spaces.

المستخلص

درسنا الفضاءات التبولوجية الغائمة. أوضحنا بعض التبولوجيات والخصائص الجبرية لبعض المجموعات الجزئية التبولوجية الغائمة. تحصلنا على بعض الخصائص التبولوجية للفضاءات غير الخطية الغائمة. أوردنا بعض التطبيقات للخصائص التبولوجية للفضاءات الخطية غير العادية الغائمة.

Table of Contents

Chapter 1

On Fuzzy Topological Spaces and Finite Dimensional Intuitionistic Fuzzy Normed Linear

Section (1-1) On Fuzzy Topological Spaces

Introduction(1.1.1)[2] After Zadeh introduced the concept of fuzzy sets, Chang developed the theory of fuzzy topological spaces, based on Zadeh's concept. From then on, quite a number of research have been published dealing with various aspects of such spaces. Gantner, Steinlage, and Warren have defined the Hausdorff separation axiom as well as subspace topology for only crisp points and crisp subsets of fuzzy spaces. Again, subspace topology has been defined by Foster for fuzzy subsets as well. We have defined the Hausdorff separation axiom as well as some of the other separation axioms by taking the fuzzy elements also into consideration. It is seen that the definitions essentially agree with these in the crisp case. Since the definition of compactness is not quite meaningful in our Hausdorff spaces, we have defined proper compactness and have shown that in a Hausdorff space (i) properly compact sets as well as sets of finite supports are closed and (ii) supports of properly compact sets are finite, if the countable union of closed sets is closed, in addition. We recall that Hutton also has defined separation axioms in fuzzy topological spaces, but his approach is entirely different from what we have adopted here. F-k spaces have been defined in terms of subspaces (as in general topology) and hereditariness of some of the separation properties are established, where Foster’s definition of subspace topology has been followed. A few examples have been provided to indicate differences between fuzzy topology and ordinary topology.

Preliminaries(1.1.2)[2] We give a few definitions valid in fuzzy spaces, while some others are included in the relevant sections. For those not given anywhere.

Let X be a set of points $\{x: x \in X\}$. A fuzzy set A in X is characterized by a membership function ,u,., from X to [0, 1], while a real subset of X (also called a crisp set) is identified with its characteristic function.

A fuzzy point or a fuzzy singleton p in X is a fuzzy set with membership function, μ_p, defined by, $\mu_p(x) = y$, for $x = x_0$, $= 0$ otherwise

where $y \in (0, 1)$. x_0 is called the support of p and y its value.

Also, p is in a fuzzy set A or $p \in A$ if and only if, $\mu_p(x_0) < \mu_A(x_0)$.

So $p \notin A \Leftrightarrow \mu_p(x_0) \geq \mu_A(x_0)$.

A real point x_1 is called a crisp point and is identified with its characteristic function. x, belongs to the fuzzy set A if, $\mu_p(x_1) = 1$.

By points (subsets) of X, we mean both crisp and fuzzy points (subsets).

The fuzzy topological space (X, τ) is as usual written as fts.

Definition (1.1.3)[2] Let (X, τ) be an fts, Y a set of points, and f: X + Y a surjection. The F-quotient topology for Y is the fuzzy topology whose open fuzzy sets are $\{B: f^{-1}[B] \in \tau)$. If $f: X \longrightarrow Y$ is an F-continuous surjection of X to Y and Y has the F-quotient topology U, then $f: (X, \tau) \longrightarrow (Y, U)$ is called an F-quotient map.

We have always assumed that the support of the fuzzy point p is x_p, unless stated otherwise. R and N as usual denote respectively the set of real numbers and the set of natural numbers.

Definition (1.1.4)[2] An fts (X, τ) is defined to be Hausdorff or $F - T_2$ if and only if the following conditions hold:

If p, q are any two points in X, then (I) if $x_p \neq x_q$ there exist open sets V, and V_q, such that $p \in V_P$, $q \notin \overline{V}_p$ and $q \in V_q$, $p \notin \overline{V}_p$;

(II) if $x_p = x_q$ and $\mu_p(x_0) < \mu_q(x_0)$, then there exists an open set $\overline{V}_p$ such that $p \in V_p$, but $q \notin \overline{V}_p$.

So if (X, t) is Hausdorff, then (I) follows immediately.

Example(1.1.5)[2] Let $(I_\alpha, (\alpha \in A)$ be the usual interval base of the relative topology on [0, 1] induced by R. We define a fuzzy topology on [0, 1] generated by the base consisting of ϕ, X and

$$\{I_{\alpha\beta}, \alpha \in A, \beta \in (0,1]\},$$

where

$$\mu_{I_{\alpha\beta}}(x) = \beta \qquad \text{for all} \quad x \in I_\alpha = 0 \text{ otherwise.}$$

Then (X, τ) is a Hausdorff fts, which is not discrete.

Definition (1.1.6)[2] A set A is open if and only if for each point $p \in A$, there exists an open set G, such that $p \in G \subset A$.

Definition (1.1.7)[2] An fts (X, τ) is F-T_1 if and only if singletons are closed.

Theorem (1.1.8)[2] An F-T_1-space is an F-T_2-space.

Proof. Let p be a fuzzy point in X. Then any point $q \in \{p\}'$ belongs to an open set V_q such that $\mu_{\{p\}'}(x_p) \geq \mu_{\overline{V}_q}(x_p)$. So $V_q \subset \{p\}'$.

If, on the other hand, p is crisp, let $x_q \in X - \{x_p\}$ be arbitrary. If $\{q_n, n \in N\}$ be a sequence of fuzzy points, where $x_{q_n} = x_{q_n}$ for all $n \in N$ and the sequence $\{\mu_{q_n}(x_q), n \in N\}$ is decreasing and converges to zero, Then there exists a sequence of open sets $\{V_{pq}, n \in N\}$, such that $p \in V_{pq}$, and $q_n \notin \overline{V}_{pq}$, for all $n \in N$, as (X, τ) is Hausdorff.

Therefore, if $P = \cap_{n \in N} \overline{V}_{pq}$, then P is a closed set, where $\mu_p(x_q) = 0$ and $\mu_p(x_p) = 1$

So P' is an open set contained in {p}' and containing the crisp point q and hence any fuzzy point with support. The definition of compactness does not seem very natural in an fts, especially when it is Hausdorff, as is shown by the following proposition.

Proposition(1.1.9)[2] No subset of a Hausdorff- fts can be compact (countably compact).

Proof. Let A be a subset of the f_{t_s} (X, τ) such that $\mu_A(x_A) > 0$, for some $x_A \in X$. Choose a sequence $\{p_n, n \in N\}$ of fuzzy points, each having support x_A, such that $\mu_{pn}(x_A) < \mu_A(x_A)$ for all n and $\{\mu_{p_n}(x_A), n \in N\}$, is an increasing sequence which converges to $\mu_A(x_A)$.

Then from the Hausdorff property, there exists a sequence of open sets,

$$\{V_{x_{A_n}},\ n \in N\}, \text{ where } p_n \in V_{x_{A_n}} \text{ and } p_{n+1} \in \overline{V}_{x_{A_n}}.$$

This sequence together with the complement of the crisp point at x_A forms an open cover of A, which has no finite subcover.

Corollary(1.1.10)[2] Singletons in an F-T_2-space are not compact (countably compact).

We therefore define open cover of a set in a slightly different way, so that the compactness arising from this definition is more meaningful in our Hausdorff fts.

Definition (1.1.11)[2] A collection $U = \{V_\alpha, a \in \Lambda, V_\alpha \in \tau\}$ is said to be a proper open cover of the set A in the fts (X, τ) if and only if for each $x \in X$, there exists $V_{\alpha x} \in U$, such that $\mu_{\alpha_x}(x) > \mu_A(x)$, U is a countable (finite) proper open cover if Λ is countable (finite). A subcollection U, of U is a proper open subcover of U if it is a proper open cover of A in its own right. Clearly, a proper cover of A is always a cover of A, but not conversely.

Definition (1.1.12)[2] A set A is properly (countably) compact in the fts (X, τ) if and only if every (countable) proper open cover of A has a proper open finite subcover. It can be seen that the crisp set (X, τ) is l-compact in the sense if and only if it is properly compact.

Proposition(1.1.13)[2] Every singleton (hence a subset with finite support) in an fts is properly compact.

Proposition (1.1.14)[2] Let (X, τ) be an fts and A a properly compact set in X. If f: (X, τ) + (Y, U) is an F-continuous subjection, then f [A] is a properly compact set in Y.

Theorem(1.1.15)[2] A properly compact set in an F-T_2-space is closed.

Proof: Let A be a properly compact set in (X, τ) and p a point in X, such that

$$\mu_p(x_p) > \mu_A(x_p). \tag{1}$$

Then, by the Hausdorff property, there exists $V_p \in \tau$ such that

$$\mu_A(x_p) < \mu_p(x_p). \tag{2}$$

and

$$\mu_p(x_p) \geq \mu_{\overline{V}_p}(x_p). \tag{3}$$

Therefore, to each point p satisfying (l), there corresponds a collection of open sets $\{V_{pq}, x_q \in X\}$, such that $\mu_A(x_q) > \mu_{V_{pq}}(x_q)$. for all $x_q \in X$

(if, however, $\mu_A(x_q) = 1$, then we must have

$$\mu_A(x_q) = \mu_{V_{pq}}(x_q). \tag{4}$$

and

$$\mu_p(x_p) \geq \mu_{V_{pq}}(x_p). \text{ for all q: } x_q \in X. \tag{5}$$

Hence

$$\mu_A(x_q) \leq \bigvee_{x_q \in X} \mu_{V_{pq}}(x_q)$$

$$\Rightarrow A \subset \bigcup_{x_q \in X} V_{pq} \Rightarrow A \bigcup_{\substack{x_{q_k} \in X \\ k=1,\dots,n}} V_{pq_k} \qquad \text{as A is proper compact}$$

$$\Rightarrow A \subset \bigcup_{k=1,..,n} \overline{V}_{pq_k} = F_p \qquad \text{say (so that } F_p \text{ is closed)}$$

$$\Rightarrow \mu_A(x_q) \le \mu_{F_p}(x_q) \qquad \text{for all } x_q \in X \tag{6}$$

This is of course accompanied by

$$\mu_p(x_p) \ge \mu_{F_p}(x_p). \tag{7}$$

Now taking into consideration all points p satisfying (l), we get the collection $\{F_p, F'_p \in t\}$, such that (6) and (7) hold. Then clearly $\mu_A(x_q) = \wedge_p \mu_{F_p}(x_q)$, for all $x_q \in X$ by (6) and (7). Hence $A = \cap_p F_p \Rightarrow A$ is closed.

Definition (1.1.16)[2] (X, τ) is a fuzzy P-space if the countable union of closed sets is closed.

Theorem(1.1.17)[2] Fuzzy Lindelof sets in a Hausdorfl fuzzy P-space are closed.

Proof. Proceeding as in Theorem (1.1.8), the open cover $(V_{pq}, x_q \in X)$ will have a countable subcover $\{Vpq_n, n \in N)$ and then $F_p = \cup_{n \in N} \overline{V}_{pq_n}$ will be a closed set.

The rest of the proof is exactly the same as in Theorem (1.1.17).

As in Gillman and Jersion, we have the following corresponding theorem for fuzzy P-spaces.

Theorem(1.1.18)[2] Properly compact sets in an F-T_1, P-space have finite supports.

Proof: Let A be a set in the F-T_1, P-space (X, τ), having at least a countable support $\{x_1, ..., x_n, ...\}$.

The set F_r defined by $\mu_{F_r}(x) = 1,\ x = x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n, \dots = 0$ otherwise

Is closed by Theorem (1.1.18) and because (X, τ) is a P-space.

If now $G_r = F'_r$ then $\{G_n, n \in N)$ is a proper open cover of A, which has no finite subcover. Hence the result follows.

Corollary(1.1.19)[2] Properly compact sets in a Hausdorff fuzzy P-space have finite supports. In ordinary topology, a closed subset of a compact (countably compact, Lindelof) set is compact (countably compact, Lindelof). The following example shows that such may not be the case in fts's, whichever definition of open cover we may follow.

Example(1.1.20)[2] Let X be an uncountable set of points and r the fuzzy topology generated by the base formed by

$$\phi,\ X,\ \{A_\alpha, \alpha \in [\tfrac{1}{4}, \tfrac{4}{5}]\} \text{ and } \{B_x, x \in X\},$$

where $\mu_{A\alpha}(x) = a$ for all $x \in X$ and

$$\mu_{B_x}(x) = \tfrac{1}{4}, = 0 \quad \text{otherwise.}$$

Then X is (properly) compact, (properly) countable compact, and Lindelof. But the (proper) open cover $\{B_x, x \in X\}$ of the closed set $A_{1/4}$ has no countable (proper) open subcover. So $A_{1/4}$ is neither (properly) compact nor Lindelof.

Again, let X be partitioned into a countable number of subsets $(X_n, n \in N\}$. Then $\{B_n$, n E N and $B_n = U_{x\in Xn} B_x\}$ is a countable (proper) open cover of $A_{1/4}$ which has no (proper) finite subcover. So $A_{1/4}$ is not (properly) countably compact.

We now consider some of the other separation properties of an fts and analyze the interrelations among them.

Definition (1.1.21)[2] (X, τ) is regu1ar (normal) if and only if for each point $p \in X$ (closed set K in X) and $V \in \tau$, where $p \in V(K \subset A)$ there exists $G \in \tau$ such that $p \in G \subset \overline{G} \subset V$ $(K \subset G \subset \overline{G} \subset V)$.

In the above, we have retained the definition of a normal.

Definition(1.1.22)[2] (X, τ) is F-T_3 (F-T_4) if and only if it is F-T_1 and regular (normal).

Theorem(1.1.23)[2] An F-T_3-space is an F-T_2-space.

Proof Let p, q be two fuzzy points, where $x_p \neq x_q$ and let w be a third fuzzy point, where $x_w = x_p$ and $\mu_w(x_p) > 1 - \mu_p(x_p)$.

Then{w}' is open and $\mu_{\{w\}'}(x) = 1 - \mu_w(x_p) < \mu_p(x_p)$ for x=x_p=1 otherwise.

Therefore q $\in$ {w}', but p $\notin$ {w}'. Now since (X, T) is regular, there exists $V_q \in \tau$ such that $q \in V_q \subset \overline{V}_q \subset \{w\}'$. Obviously then, $p \notin \overline{V}_q$.

Similarly, an open set V_p can be determined such that $p \in V_p$ and $q \notin \overline{V}_q$.

The other cases can be similarly handled.

Theory (1.1.24)[2] An F-T_4-space is an F-T_3-space. The converse results (as usual) are not true in general.

Remark(1.1.25)[2] It can be easily seen that the fuzzy space of Example(1.1.20) is an F-T_4-space.

Fuzzy Subspace(1.1.26)[2]

We first define subspace topology on a subset of X, following.

Definition (1.1.27)[2] Let A be a subset of the fts (X, τ). The collection

$$\tau_A = \{V_A, V_A = V \cap A, V \in \tau\}$$

constitutes a fuzzy topology on A. Consequently, (A, τ_A) is called a fuzzy subspace of (X, τ) and V, is an open subset of A in τ_A.

Remark(1.1.28)[2] The subspace topology defined for crisp subsets of X essentially agrees with this definition.

Definition (1.1.29)[2] Let A be a subset of X. Then the set B, where $\mu_B(x) = \mu_A(x) - (\mu_A \wedge \mu_A)(x)$, f or some $V_{E\,t}$ and for all x $\in$ X, is called a closed subset of A in τ_A.

Remark(1.1.30)[2] It can be shown without much difficulty that closed subsets of A are not obtained from those of X, as they are in ordinary topology, which again is a variation of fuzzy topology from ordinary topology.

Definition(1.1.31)[2] A property P in an fts (X, τ) is said to be hereditary if it is satisfied by each subset of X. The following theorem shows that at least one of the separation properties is hereditary.

Theorem(1.1.32)[2] $F - T_1$, is a hereditary property.

Proof Let A be a fuzzy set in (X, τ) and p a fuzzy point in A. Let (X, τ) be F-T_1. If w be the fuzzy point, where $x_w = x_p$ and

$$\mu_A(x_p) - \mu_{Ap}(x_p) = 1 - \mu_w(x_p),$$

then, $(\mu_A \wedge \mu_{\{w\}'})(x) = 1 - \mu_w(x_p)$ for $x = x_p = \mu_A(x)$ otherwise So

$$[\mu_A - \mu_A \wedge \mu_{\{w\}'}](x) \ = \mu_p(x_p) \text{ for } \ x = x_p \ = 0 \ \text{ otherwise.}$$

Hence {p} is closed in τ_A.

If, on the other hand, p is crisp and A is fuzzy (or crisp), then

$[\mu_A - \mu_A \wedge \mu_{\{w\}'}](x) \ = \mu_p(x_p)$ for all $x \in X$.

This completes the proof.

Definition (1.1.33)[2] The fts (X, τ) is an F-k-space if and only if any subset A of X is open if and only if A ∩ C is open in τ_C, for each compact set C in X.

Christoph, however, has defined (X, r) to be an F-k-space if a fuzzy set A is closed if and only if A n C is closed in X, for each compact fuzzy set C in X.

Theorems (1.1.23) and (1.1.32) depend for their proofs on the arguments that (i) if A is a fuzzy set and p a fuzzy point where $p \in A'$ then $p \in A$, and (ii) if A is closed, then no $p \in A'$ can be such that every $V \in \tau$ where

$p \in V$, satisfies the condition $V \cap A \neq \phi$. The following example shows that these arguments are generally not true.

Example(1.1.34)[2] We take $\alpha = \frac{1}{4}$. Each fuzzy point with value $< \frac{1}{3}$ belongs to $A'_{1/3}$, although it belongs to $A_{1/3}$ as well, which contradicts (i). Again, $A_{1/3}$ is a closed set. But each fuzzy point with value $< \frac{2}{3}$ and $> \frac{1}{3}$ belongs to $A'_{1/3}$ (and not to $A_{1/3}$) and is such that every open set containing the point intersects $A_{1/3}$. This contradicts (ii).

Remark(1.1.35)[2] It is interesting to note in this connection that the converse result, namely, if A contains all its accumulation points, then it is closed, is not always true, which can be easily shown by examples.

It is however observed that even with our changed definition of an F-k space. Theorem (1.1.36), is true, if we follow Christoph's definition of local compactness.

Theorem(1.1.36)[2] Let(X,τ) be a locally compact fts and

$f: (X, \tau) \longrightarrow (Y, D)$ an F-quotient map. Then (Y, D) is an F-k-space.

Proof , for ordinary topology. However, we give a complete form of the proof as applied to fuzzy spaces.

Let A be a subset of Y such that $A \cap C$ is open in τ_C for each compact set C in Y. Since (X, τ) is locally compact, there exists $U \in r$ such that $U \subset V$ where V is compact in X. So f [V] is compact in Y.

$$\Rightarrow A \cap f[V] \text{ is open in } \tau_{f[V]}$$

$$\Rightarrow A \cap f[V] = f[V] \cap G \text{ for some } G \in D.$$

Therefore, $f^{-1}[A] \cap f^{-1}(f[V] = f^{-1}(f[V])) \cap f^{-1}[G]$. Taking intersection with U on both sides,

$$U \cap f^{-1}[A] = U \cap f^{-1}[G] \text{ as } U \subset f^{-1}(f[V])$$

$$\Rightarrow U \cap f^{-1}[A] \text{ is open in } (X, \tau).$$

Now X= ∪ {U, U ∈ τ and U ⊂ some compact set V of X} as (X, τ) is locally compact. Therefore

$f^{-1}[A] = \cup\{U \cap f^{-1}[A], U \in \tau$ and $U \subset$ some compact set V of X}= open in X⇒A is open in (Y, D), as f is an F-quotient map.

Corollary(1.1.37)[2] (Christoph). A locally compact fts is an F-k-space.

Definition(1.1.38)[2] An fts (X, τ) is locally Lindelof if and only if each point in X has a Lindelof neighbourhood.

Definition (1.1.39)[2] (X, τ) is an F--space if and only if a set V is open if and only if V ∩ L is open in τ_L, for each Lindelof set L of X.

Since F-continuous image of a Lindelof set is always Lindelof, a result analogous to Theorem (1.1.36) can be stated as follows.

Theorem(1.1.40)[2] Let (X, τ) be a locally Lindeloffls and f: (X, τ) ⟶ (Y, U) an F-quotient map. Then (Y, U) is an F-ω_L-space.

Corollary (1.1.41)[2] A locally Lindelof fts is an F-ω_L-space.

Corresponding to the properly compact sets in (X, τ), a concept similar to F - k-space can be defined as follows:

Definition (1.1.41)[2] An fts (X, τ) is a properly F-k-space if a set A in it is open if and only if A ∩ C is open in τ_C, for each properly compact set C in X.

Section (1-2) Finite Dimensional Intuitionistic Fuzzy Normed Linear

T. Bag and S. K. Samanta introduced the definition of fuzzy norm over a linear space following the definition S. C. Cheng and J. N. Moordeson and they have studied finite dimensional fuzzy normed linear spaces. Also the definition of intuitionistic fuzzy n-normed linear space was introduced and established a sufficient condition for an intuitionistic fuzzy n-normed linear space to be complete. Following the definition of intuitionistic fuzzy n-norm, the definition of intuitionistic fuzzy norm (in short IFN) is defined over a linear space. There after a sufficient condition is given for an intuitionistic fuzzy normed linear space to be complete and also it is proved that a finite dimensional intuitionistic fuzzy norm linear space is complete. In such spaces, it is established that a necessary and sufficient condition for a subset to be compact. Thereafter following the definition of fuzzy continuous mapping, the definition of intuitionistic fuzzy continuity, strongly intuitionistic fuzzy continuity and sequentially intuitionistic fuzzy continuity are defined and proved that the concept of intuitionistic fuzzy continuity and sequentially intuitionistic fuzzy continuity are equivalent. There after it is shown that intuitionistic fuzzy continuous image of a compact set is again a compact set.

Definition(1.2.1)[25] A binary operation $*$: $[0, 1] \times [0,1] \longrightarrow [0, 1]$ is continuous t - norm if $*$ satisfies the following conditions :

(i) $*$ is commutative and associative ,

(ii) $*$ is continuous ,

(iii) $a * 1 = a\ \forall\ a\ \varepsilon\ [0, 1]$,

(iv) $a * b \le c * d$ whenever $a \le c$, $b \le d$ and $a , b , c , d \in [0 , 1]$.

Definition(1.2.2)[25]A binary operation $\Diamond$: $[0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-co-norm if $\Diamond$ satisfies the following conditions :

(i) $\Diamond$ is commutative and associative ,

(ii) $\Diamond$ is continuous ,

(iii) $a \Diamond 0 = a\ \forall\ a$ s $[0, 1]$,

(iv) $a \Diamond b < c \Diamond d$ whenever $a \le c$, $b \le d$ and $a , b , c , d \in [0,1]$.

Remark(1.2.3)[25] (a) For any $r_1, r_2\ \varepsilon\ (0, 1)$ with $r_1 > r_2$, there exist $r_3 , r_4\ \varepsilon\ (0, 1)$ such that $r_1 * r_3 > r_2$ and $r_1 > r_4 \Diamond r_2$.
(b) For any $r_5\ \varepsilon\ (0,1)$, there exist r_6, r_7 s $(0,1)$ such that $r_6 * r_6 \ge r_5$ and $r_7 \Diamond r_7 \le r_5$.

Definition(1.2.4)[25] Let E be any set. An intuitionistic fuzzy set A of E is an object of the form A = { (x , μ_A(x) , ν_A(x)) : x $\in$ E } , where the functions μ_A: E$\longrightarrow$ [0, 1] and ν_A: E $\longrightarrow$ [0,1] denotes the degree of membership and the non -membership of the element x$\in$E respectively and for every x $\in$ E , $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition(1.2.5)[25] If A and B are two intuitionistic fuzzy sets of a on- empty set E , then A $\subseteq$ B if and only if for all x ε E,

$$\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x);$$

A = B if and only if for all x ε E ,

$$\mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x) ;$$

$$\overline{A} = \{(x , \nu_A(x) , \mu_A(x)) : x \,\varepsilon\, E \} ;$$

$$A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max (\nu_A(x), \nu_B(x))): x \,\varepsilon\, E\};$$

$$A \cap B = \{(x, \max(\mu_A(x), \mu_B(x)), \min (\nu_A(x), \nu_B(x))): x \,\varepsilon\, E\};$$

Definition(1.2.6)[25] Let $*$be a continuous t-norm, ◊ be a continuous t- co-norm and V be a linear space over the field F (= R or C). An Intuitionistic fuzzy norm or in short IFN on V is an object of the form A={((x,t), N(x, t), M (x, t)): (x, t) ε V × R^+ }, where N , M are fuzzy sets on V × R^+ , N denotes the degree of membership and M denotes the degree of non -membership (x, t) ε V × R^+ satisfying the following conditions :

(i) $N(x, t) + M(x, t) \leq 1 \ \forall\ (x, t) \,\varepsilon\, V \times R^+$;

(ii) $N(x, t) > 0$;

(iii) N(x, t) = 1 if and only if x = $\underline{0}$;

(iv) $N(c\, x, t) = N(x, \frac{t}{|c|})$ $c \neq 0$, c ε F ;

(v) $N(x, s) * N(y, t) \leq N\, x + y, s + t)$;

(vi) N (x , ·) is non - decreasing function of R^+ and $\lim_{t\to\infty} N(x, t)=1$;

(vii) M (x, t) > 0 ;

(viii) M (x, t) = 0 if and only if x = $\underline{0}$;

(ix) M (c x, t) = M (x , $\frac{t}{|c|}$) c ≠ 0 , c ε F ;

(x) M (x, s) ◊ M (y, t) ≥ M (x + y, s + t) ;

(xi) M (x, ·) is non - increasing function of R^+ and $\lim_{t\to\infty}$ M (x, t) = 0.

Example(1.2.7)[25] Let (V= R, ||·||) be a normed linear space where||x||=|x| ∀ x ε R. Define a ∗b=min {a, b}and a ◊ b =max{a, b} for all a , b ε [0, 1] . Also define N (x, t) = $\frac{t}{t+k|x|}$ and M (x, t) = $\frac{k|t|}{t+k|x|}$ where k > 0. We now consider

A={((x, t), N(x, t), M(x, t)): (x, t) ε V× R^+}.Here A is an IFN on V.

Definition(1.2.8)[25] If A is an IFN on V (a linear space over the field F(= R or C)) then (V, A) is called an intuitionistic fuzzy normed linear space or in short IFNLS.

Definition(1.2.9)[25] A sequence $\{x_n\}_n$ in an IFNLS (V, A) is said to converge to x ε V if given r > 0 , t > 0 , 0 < r < 1 there exists an integer n_0 ε N such that $N(x_n - x, t) > 1 - r$ and

$M(x_n - x, t) < r$ for all $n \geq n_0$.

Theorem(1.2.10)[25] If a sequence $\{x_n\}_n$ in an IFNLS (V, A) is convergent, it's limit is unique.

Proof: Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Also let s , t ε R+.

Now,

$$\lim_{n\to\infty} x_n = x \Rightarrow \begin{cases} \lim_{n\to\infty} N(x_n - x, t) = 1 \\ \lim_{n\to\infty} M(x_n - x, t) = 0 \end{cases}$$

$$\lim_{n\to\infty} x_n = y \Rightarrow \begin{cases} \lim_{n\to\infty} N(x_n - y, t) = 1 \\ \lim_{n\to\infty} M(x_n - y, t) = 0 \end{cases}$$

$$N(x - y, s + t) = N(x - x_n + x_n - y, s + t)$$
$$\geq N(x - x_n, s) * N(x_n - y, t)$$
$$= N(x_n - x, s) * N(x_n - y, t)$$

Taking limit, we have

$$N(x - y, s + t) \geq \lim_{n\to\infty} N(x_n - x, s) * \lim_{n\to\infty} N(x_n - y, t) = 1$$

$$\Rightarrow N(x - y, s + t) = 1 \Rightarrow x - y = \underline{0} \Rightarrow x = y$$

Theorem(1.2.11)[25] If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} x_n + y_n = x + y$ in an IFNLS (V, A) .

Proof: Let s , t ε R^+. Now,

$$\lim_{n\to\infty} x_n = x \Rightarrow \begin{cases} \lim_{n\to\infty} N(x_n - x, t) = 1 \\ \lim_{n\to\infty} M(x_n - x, t) = 0 \end{cases}$$

$$\lim_{n\to\infty} y_n = y \Rightarrow \begin{cases} \lim_{n\to\infty} N(y_n - y, t) = 1 \\ \lim_{n\to\infty} M(y_n - y, t) = 0 \end{cases}$$

Now,

$$N((x_n + y_n) - (x + y), s + t) = N((x_n - x) + (y_n - y), s + t)$$
$$\geq N(x_n - x, s) * N(y_n - y, t)$$

Taking limit, we have

$$\lim_{n\to\infty} N((x_n + y_n) - (x + y), s + t)$$
$$\geq \lim_{n\to\infty} N(x_n - x, s) * N(y_n - y, t) = 1 * 1 = 1$$
$$\lim_{n\to\infty} N((x_n + y_n) - (x + y), s + t) = 1$$

Again,

$$M((x_n + yn) - (x + y), s + t) = M((x_n - x) + (y_n - y), s + t)$$
$$\leq M(x_n - x, s) \lozenge M(y_n - y, t)$$

Taking limit, we have

$$\lim_{n\to\infty} M((x_n + y_n) - (x + y), s + t) \le \lim_{n\to\infty} M(x_n - x, s) \lozenge \lim_{n\to\infty} M(y_n - y, t)$$

$$= 0 \lozenge 0 = 0$$

$$\Rightarrow \lim_{n\to\infty} M((x_n + y_n) - (x + y), s + t) = 0$$

Thus, we see that $\lim_{n\to\infty} x_n + y_n = x + y$.

Definition(1.2.12)[25] A sequence $\{x_n\}_n$ in an IFNLS (V, A) is said to be a Cauchy sequence if $\lim_{n\to\infty} N(x_n{+}p{-}x_n,t) = 1$ and

$$M(x_n{+}p{-}x_n, t)=0,\ p = 1, 2, 3, \cdots, t > 0.$$

Theorem(1.2.13)[25] In an IFNLS (V,A), every convergent sequence is a Cauchy sequence.

Proof: Let $\{x_n\}_n$ be a convergent sequence in the IFNLS (V, A) with $\lim_{n\to\infty} x^n = x$. Let s, t ε R^+ and p = 1 , 2 , 3 , $\cdots$, we have

$$N(x_n + p - x_n, s + t) = N(x_n + p - x + x - x_n, s + t)$$

$$\ge N(x_n + p - x, s) * N(x - x_n, t)$$

$$= N(x_n + p - x, s) * N(x_n - x, t)$$

Taking limit , we have

$$\lim_{n\to\infty} N(x_{n+p} - x_n, s + t)$$

$$\ge \lim_{n\to\infty} N(x_{n+p} - x, s) * \lim_{n\to\infty} N(x_n - x, t) = 1 * 1 = 1$$

$$\Rightarrow \lim_{n\to\infty} N(x_{n+p} - x_n, s + t) = 1\ \forall\ s, t\ \varepsilon\ R^+ \text{ and } p = 1, 2, 3, \cdots$$

Again,

$$M(x_{n+p} - x_n, s + t) = M(x_{n+p} - x + x - x_n, s + t)$$

$$\le M(x_{n+p} - x, s) \lozenge M(x - x_n, t)$$

$$= M(x_{n+p} - x, s) \lozenge M(x_n - x, t)$$

Taking limit, we have

$$\lim_{n\to\infty} M(x_{n+p} - x_n, s + t)$$

$\leq \lim_{n\to\infty} M(x_{n+p} - x, s) \lozenge \lim_{n\to\infty} M(x_n - x, t) = 0 \lozenge 0 = 0$

$$\Rightarrow \lim_{n\to\infty} M(x_{n+p} - x_n, s + t) = 0 \ \forall \ s, t \ \varepsilon \ R^+ \text{ and } p = 1, 2, 3, \cdots$$

Thus, $\{x_n\}_n$ is a Cauchy sequence in the IFNLS (V, A) .

Section(1-3) The converse of the above theorem is not necessarily true

Example(1.3.1)[25] Let (V, $\|\cdot\|$) be a normed linear space and define $a * b = \min\{a, b\}$ and $a \lozenge b = \max\{a, b\}$ for all a, b ε [0, 1]. For all t > 0, define $N(x, t) = \frac{t}{t+k|x|}$ and $M(x, t) = \frac{k\|x\|}{t+k|x|}$ where k > 0 .

It is easy to see that A = {((x, t), N (x, t), M (x, t)):(x, t) ε V × R+} is an IFN on V. We now show that

(a) $\{x_n\}_n$ is a Cauchy sequence in (V, $\|\cdot\|$) if and only if $\{x_n\}_n$ is a Cauchy sequence in the IFNLS (V, A) .

(b) $\{x_n\}_n$ is a convergent sequence in (V, $\|\cdot\|$) if and only if $\{x_n\}_n$ is a convergent sequence in the IFNLS (V, A) .

Proof:

(a) Let $\{x_n\}_n$ be a Cauchy sequence in (V, $\|\cdot\|$) and t > 0.

$$\Leftrightarrow \lim_{n\to\infty} \|x_n + p - x_n\| = 0 \text{ for } p = 1, 2, \ldots$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{t}{t + k\|x_{n+p} - x_n\|} = 1 \text{ and } \lim_{n\to\infty} \frac{k\|x_{n+p} - x_n\|}{t + k\|x_{n+p} - x_n\|} = 0$$

$$\Leftrightarrow \lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1 \text{ and } \lim_{n\to\infty} M(x_{n+p} - x_n, t) = 0$$

$$\Leftrightarrow \{x_n\}_n \text{ is a Cauchy sequence in (V, A)}$$

(b) Let $\{x_n\}$n be a convergent sequence in (V, $\|\cdot\|$) and t > 0 .

$$\Leftrightarrow \lim_{n\to\infty} \| x_n - x \| = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{t}{t+k\|x_{n+p}-x_n\|} = 1 \text{ and } \lim_{n\to\infty} \frac{k\|x_{n+p}-x_n\|}{t+k\|x_{n+p}-x_n\|} = 0$$

$$\Leftrightarrow \lim_{n\to\infty} N\,(x_n - x, t) = 1 \text{ and } \lim_{n\to\infty} M\,(x_n - x, t) = 0$$

$\Leftrightarrow\{x_n\}_n$ is a convegent sequence in (V, A).

Theorem(1.3.2)[25] Let (V, A) be an IFNLS, such that every Cauchy sequence in (V, A) has a convergent subsequence. Then (V, A) is complete .

Proof: Let $\{x_n\}_n$ be a Cauchy sequence in (V, A) and $\{x_{nk}\}_k$ be a subsequence of $\{x_n\}_n$ that converges to x ε V and t > 0 . Since $\{x_n\}_n$ is a Cauchy sequence in (V, A) , we have

$$\Leftrightarrow \lim_{n,k\to\infty} N\left(x_n - n_k, \frac{t}{2}\right) = 1 \text{ and } \lim_{n,k\to\infty} M\left(x_n - n_k, \frac{t}{2}\right) = 0$$

Again since $\{x_{nk}\}_k$ converges to x , we have

$$\lim_{n,k\to\infty} N\left(x_n - n_k, \frac{t}{2}\right) = 1 \text{ and } \lim_{n,k\to\infty} M\left(x_n - n_k, \frac{t}{2}\right) = 0$$

Now,

$$N\,(x_n - x, t) = N\,(x_n - x_{nk} + x_{nk} - x, t) \geq$$

$$N\left(x_n - n_k, \frac{t}{2}\right) * M\left(x_n - n_k, \frac{t}{2}\right)$$

$$\Leftrightarrow \lim_{n\to\infty} N\,(x_n - x, t) = 1$$

Again, we see that

$$M\,(x_n - x, t) = M\,(x_n - x_{nk} + x_{nk} - x\,,\, t\,)$$

$$\leq M\left(x_n - n_{n_k}, \frac{t}{2}\right) \Diamond M\left(x_{n_k} - x, \frac{t}{2}\right) \Leftrightarrow \lim_{n\to\infty} M\,(x_n - x, t) = 0$$

Thus, $\{x_n\}_n$ converges to x in (V, A) and hence (V, A) is complete.

Theorem (1.3.3)[25] Let (V, A) be an IFNLS , we further assume that,

(xii) $\left.\begin{array}{l} a \Diamond a = a \\ a * a = a \end{array}\right\} a \quad \forall\, a\ \varepsilon\ [0, 1]$

(xiii) N (x, t) > 0 $\forall$ t > 0 $\Rightarrow$ x = 0

(xiv) M (x, t) > 0 $\forall$ t > 0 $\Rightarrow$ x = $\underline{0}$

Define $\|x\|_\alpha^1 = \wedge$ {t :N (x, t)$\geq\alpha$} and $\|x\|_\alpha^2$ = $\vee$ {t :M (x, t) $\leq \alpha$, α ε (0,1).

Then both {$\|x\|_\alpha^1$: α ε (0, 1)} and {$\|x\|_\alpha^2$: α ε (0, 1)} are ascending family of norms on V. We call these norms as α - norm on V corresponding to the IFN A on V .

Proof: Let α ε (0, 1) . To prove $\|x\|_\alpha^1$ is a norm on V, we will prove the followings :

(1) $\|x\|_\alpha^1 \geq 0$ 8 x ≥ V ;

(2) $\|x\|_\alpha^1 = 0 \Leftrightarrow x = 0$;

(3) $\|cx\|_\alpha^1 = |\, c\, |\ \|x\|_\alpha^1$;

(4) $\|x+y\|_\alpha^1 \leq \|x\|_\alpha^1 + \|y\|_\alpha^1$.

The proof of (1), (2),(3) and (4) directly follows from the proof of the theorem (1.3.2) . So, we now prove (4).

$$\|x\|_\alpha^1 + \|y\|_\alpha^1 = \wedge\{s: N (x, s) \geq \alpha\} + \wedge\{t: N (y, t) \geq \}$$

= $\wedge$ {s + t: N(x, s) ≥α, N(y, t) ≥ α} = $\wedge$ {s + t: N(x, s)$*$ N(y, t α$*$ α} ≥ $\wedge$ {s + t:N (x + y, s + t) ≥ α} = $\|x+y\|_\alpha^1$, which proves (4) .

Let $0<\alpha_1<\alpha_2<1$. $\|x\|_{\alpha_1}^1 = \wedge\{t: N (x, t) \geq \alpha_1\}$ and $\|x\|_{\alpha_2}^1 = \wedge\{t: N(x, t)\geq \alpha_2\}$.

Since $\alpha_1 < \alpha_2$, $\{t: N (x, t) \geq \alpha_2\} \subset \{t: N (x, t) \geq \alpha_1\}$

$\Rightarrow \wedge\{t: N (x, t) \geq \alpha_2\} \geq \wedge\{t: N (x, t)\geq \alpha_1\} \Rightarrow \|x\|_{\alpha_2}^1 \geq \|x\|_{\alpha_1}^1$. Thus, we see that {$\|x\|_\alpha^1$: α ε (0, 1)} is an ascending family of norms on V.

Now we shall prove that $\{\|x\|_\alpha^2 : \alpha \, \varepsilon \, (0, 1)\}$ is also an ascending family of norms on V. Let $\alpha \, \varepsilon \, (0, 1)$ and x , y ε V. It is obvious that $\|x\|_\alpha^2 \geq 0$. Let $\|x\|_\alpha^2 = 0$. Now, $\|x\|_\alpha^2 = 0 \Rightarrow \vee\{t: M(x, t) \leq \alpha\} = 0 \Rightarrow M(x, t) > \alpha > 0 \ \forall \ t > 0 \Rightarrow x = 0$. Conversely, we assume that

$x = 0 \Rightarrow M(x, t) = 0 \ \forall \ t > 0 \Rightarrow \vee\{t: M(x, t) \leq \alpha\} = 0 \Rightarrow \|x\|_\alpha^2 = 0$.

It is easy to see that k $\|cx\|_\alpha^2 = |c| \, \|x\|_\alpha^2 \ \forall \ c \, \varepsilon \, F$.

$\|x\|_\alpha^2 + \|y\|_\alpha^2 = \vee \{s: M(x, s) \leq \alpha\} + \vee\{t: M(y, t) \leq \alpha\} = \vee \{s + t: M(x, s) \leq \alpha, M(y, t) \leq \alpha\} = \vee \{s + t: M(x, s) \Diamond M(y, t) \leq \alpha \Diamond \alpha\} \geq \{s + t: M(x + y, s + \ ) \leq \alpha\} = \|x + y\|_\alpha^2$, that is $\|x + y\|_\alpha^2 \leq \|x\|_\alpha^2 + \|y\|_\alpha^2$ $\forall$ x , y ε V .

Let $0 < \alpha_1 < \alpha_2 < 1$. Therefore, $\|x\|_{\alpha_1}^1 = \vee \{t : M(x, t) \leq \alpha_1\}$ and $\|x\|_{\alpha_2}^1 = \vee \{t: M(x, t) \leq \alpha_2\}$. Since $\alpha_1 < \alpha_2$, we have

$$\{t: M(x, t) \leq \alpha_1\} \subset \{t: M(x, t) \leq \alpha_2\}$$

$$\Rightarrow \vee\{t: M(x, t) \leq \alpha_1\} \leq \vee\{t: M(x, t) \leq \alpha_2\}$$

$\Rightarrow \|x\|_{\alpha_1}^2 \leq \|x\|_{\alpha_2}^2$. Thus we see that $\{\|x\|_\alpha^2 : \alpha \, \varepsilon \, (0, 1)\}$ is an ascending family of norms on V.

Lemma(1.3.4)[25] Let (V, A) be an IFNLS satisfying the condition (Xiii) and $\{x_1, x_2, \ldots, x_n\}$ be a finite set of linearly independent vectors of V. Then for each α ε (0, 1) there exists a constant $C_\alpha > 0$ such that for any scalars $\alpha_1 , \alpha_2 , \cdot \cdot \cdot , \alpha_n$,

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\|_\alpha^1 \geq C_\alpha \sum_{i=1}^{n} |\alpha_i|$$

where $\| \cdot \|_\alpha^1$ is defined in the previous theorem.

Theorem(1.3.5)[25] Every finite dimensional IFNLS satisfying the conditions (xii) and (xiii) is complete .

Proof: Let (V, A) be a finite dimensional IFNLS satisfying the conditions (xii) and (xiii) . Also, let dim V = k and $\{e_1, e_2, \ldots, e_k\}$ be a basis of V . Consider $\{x_n\}_n$ as an arbitrary Cauchy sequence in (V, A) .

Let $x_n = \beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + \cdot\cdot\cdot + \beta_k^{(n)} e_k$ where $\beta_1^{(n)}, \beta_2^{(n)}, \ldots, \beta_k^{(n)}$ are suitable scalars. Then by the same calculation of the theorem (1.3.5) , there exist $\beta_1, \beta_2, \ldots, \beta_k \,\varepsilon\, F$ such that the sequence $\{\beta_i^{(n)}\}_n$ converges to β_i for i = 1, 2,... , k. Clearly $x = \sum_{i=1}^{k} \beta_i e_i \varepsilon V$. Now, for all t > 0,

$$N(x_n - x, t) = N\left(\sum_{i=1}^{k} \beta_i^{(n)} e_i - \sum_{i=1}^{k} \beta_i e_i, t\right)$$

$$= N\left(\sum_{i=1}^{k} \left(\beta_i^{(n)} - \beta_i\right) e_i, t\right)$$

$$\geq N\left(\left(\beta_1^{(n)} - \beta_1\right) e_1, \tfrac{t}{k}\right) * \ldots * \left(\left(\beta_k^{(n)} - \beta_1\right) e_k, \tfrac{t}{k}\right)$$

$$= N\left(e_1, \frac{t}{k\left|\beta_1^{(n)} - \beta_1\right|}\right) * \ldots * N\left(e_k, \frac{t}{k\left|\beta_k^{(n)} - \beta_k\right|}\right)$$

Since $\lim_{n\to\infty} \frac{t}{k\left|\beta_1^{(n)} - \beta_1\right|} = \infty$, we see that $\lim_{n\to\infty} N\left(e_i, \frac{t}{k\left|\beta_i^{(n)} - \beta_i\right|}\right) = 0$

$$\Rightarrow \lim_{n\to\infty} N(x_n - x, t) \geq 1 * \cdot\cdot\cdot * 1 = 1 \;\forall\; t > 0$$

$$\Rightarrow \lim_{n\to\infty} N(x_n - x, t) = 1 \;\forall\; t > 0$$

Again, for all t > 0,

$$M(x_n - x, t) = M\left(\left(\beta_1^{(n)} - \beta_1\right) e_1, \tfrac{t}{k}\right) \Diamond \ldots \Diamond \left(\left(\beta_k^{(n)} - \beta_1\right) e_k, \tfrac{t}{k}\right)$$

$$= M\left(e_1, \frac{t}{k\left|\beta_1^{(n)} - \beta_1\right|}\right) \Diamond \ldots \Diamond\, M\left(e_k, \frac{t}{k\left|\beta_k^{(n)} - \beta_k\right|}\right)$$

Since $\lim\limits_{n\to\infty} \frac{t}{k\left|\beta_1^{(n)}-\beta_1\right|} = \infty$, we see that $\lim\limits_{n\to\infty} M\left(e_i, \frac{t}{k\left|\beta_i^{(n)}-\beta_i\right|}\right) = 1$

$$\Rightarrow \lim_{n\to\infty} M\,(x_n - x, t) \le 1 \lozenge \cdots \lozenge 1 = 1 \ \forall\ t > 0$$

$$\Rightarrow \lim_{n\to\infty} M\,(x_n - x, t) = 0 \ \forall\ t > 0$$

Thus, we see that $\{x_n\}_n$ is an arbitrary Cauchy sequence that converges to x ε V , hence the IFNLS (V, A) is complete .

Definition (1.3.6)[25] Let (V, A) be an IFNLS. A subset P of V is said to be closed if for any sequence $\{x_n\}_n$ in P converges to x ε P , that is, $\lim\limits_{n\to\infty} N\,(x_n - x, t) = 1$, and $\lim\limits_{n\to\infty} M\,(x_n - x, t) = 0 \Rightarrow x \ \varepsilon\ P$.

Definition (1.3.7)[25] Let (V, A) be an IFNLS. A subset Q of V is said to be the closure of P ($\subset$ V) if for any x ε Q , there exists a sequence $\{x_n\}_n$ in P such that

$$\lim_{n\to\infty} N\,(x_n - x, t) = 1, \text{ and } \lim_{n\to\infty} M\,(x_n - x, t) = 0 \ \forall\ t\ \varepsilon\ R^+.$$

We denote the set Q by $\overline{P}$.

Definition (1.3.8)[25] Let (U, A) and (V, B) be two IFNLS over the same field F. A mapping f from (U, A) to (V, B) is said to be intuitionistic fuzzy continuous (or in short IFC) at x_0 ε U , if for any given ε > 0, α ε (0, 1) , ∃ δ = δ (α, ε) > 0, β = β (α, ε) ε (0, 1) such that for all x ε U,

$$N_U\,(x - x_0, \delta) > \beta \Rightarrow N_V\,(f(x) - f(x_0), \varepsilon) > \alpha$$

and

$$M_U\,(x - x_0, \delta) < 1 - \beta \Rightarrow M_V\,(f(x) - f(x_0)\,, \varepsilon) < 1 - \alpha\,.$$

If f is continuous at each point of U , f is said to be IFC on U .

Definition(1.3.9)[25] A mapping f from (U, A) to (V, B) is said to be strongly intuitionisticfuzzy continuous(or in short strongly IFC) at x_0 ε U , if for any given ∃ δ = δ (α, ε) > 0 such that for all x ε U,

$$N_V(f(x) - f(x_0), \varepsilon) \geq N_U(x - x_0, \delta) \text{ and}$$

$$M_V(f(x) - f(x_0), \varepsilon) < M_U(x - x_0, \delta) .$$

f is said to be strongly IFC on U if f is strongly IFC at each point of U .

Definition(1.3.10)[25] A mapping f from (U, A) to (V, B) is said to be sequentially intuitionistic fuzzy continuous (or in short sequentially IFC) at $x_0 \varepsilon U$, if for any sequence $\{x_n\}_n$, $x_n \varepsilon U \; \forall n$, with $x_n \longrightarrow x_0$ in (U, A) implies $f(x_n) \longrightarrow f(x_0)$ in (V, B), that is,

$$\lim_{n\to\infty} N_U(x_n - x_0, t) = 1 \text{ and } \lim_{n\to\infty} M_U(xn - x_0, t) = 0$$

$$\Rightarrow \lim_{n\to\infty} N_V(f(x_n) - f(x_0), t) = 1 \text{ and } \lim_{n\to\infty} M_V(f(x_n) - f(x_0), t) = 0$$

If f is sequentially IFC at each point of U then f is said to be sequentially IFC on U.

Theorem(1.3.11)[25] Let f be a mapping from (U, A) to (V, B). If f strongly IFC then it is sequentially IFC but not conversely .

Proof: Let f : (U, A) $\longrightarrow$ (V , B) be strongly IFC on U and $x_0 \varepsilon U$. Then for each $\varepsilon > 0$, $\exists \delta = \delta(x_0, \varepsilon) > 0$ such that for all $x \varepsilon U$,

$$N_V(f(x) - f(x_0), \varepsilon) \geq N_U(x - x_0, \delta) \text{ and}$$

$$M_V(f(x) - f(x_0), \varepsilon) < M_U(x - x_0, \delta)$$

Let $\{x_n\}_n$ be a sequence in U such that $x_n \longrightarrow x_0$, that is, for all $t > 0$,

$$\lim_{n\to\infty} N_U(x_n - x_0, t) = 1 \text{ and } \lim_{n\to\infty} M_U(x_n - x_0, t) = 0$$

Thus, we see that

$$N_V(f(x_n) - f(x_0), \varepsilon) \geq N_U(x_n - x_0, \delta) \text{ and}$$

$$M_V(f(x_n) - f(x_0), \varepsilon) < M_U(x_n - x_0, \delta)$$

which implies that

$$\lim_{n\to\infty} N_V(f(x_n) - f(x_0), \varepsilon) = 1 \text{ and } \lim_{n\to\infty} M_V(f(x_n) - f(x_0), \varepsilon) = 0$$

that is, $f(x_n) \longrightarrow f(x_0)$ in (V, B) .

To show that the sequentially IFC of f does not imply strongly IFC of f on U, consider the following example.

Example(1.3.12)[25] Let (X= R, ||·||) be a normed linear space where ||x|| = | x | ∀ x ε R. Define a ∗ b = min{a ,b} and a ◊ b = max{a, b}

for all a , b ε [0, 1] . Also, define

$N_1, M_1, N_2, M_2: X \times R^+ \longrightarrow [0, 1]$ by

$$N_1(x,t) = \frac{t}{t+|x|}, \quad M_1(x,t) = \frac{|x|}{t+|x|}$$

$$N_2(x,t) = \frac{t}{t+k|x|}, \quad M_2(x,t) = \frac{k|x|}{t+k|x|} \ k > 0$$

Let $A=\{((x, t), N_1, M_1): (x, t) \in X \times R^+\}$ and

$B=\{((x, t), N_2, M_2): (x, t) \in X \times R^+\}$

It is easy to see that (X, A) and (X, B) are IFNLS . Let us now define, $f(x) = \frac{x^4}{1+x^2}$ ∀ x ε X . Let x_0 ε X and $\{x_n\}_n$ be a sequence in X such that $x_n \longrightarrow x_0$ in (X, A) , that is , for all t > 0,

$$\lim_{n\to\infty} N_1(x_n - x_0, t) = 1 \text{ and } \lim_{n\to\infty} M_1(x_n - x_0, t) = 0$$

that is, $\lim_{n\to\infty} \frac{t}{t+|x_n-x_0|} = 1$ and $\lim_{n\to\infty} \frac{|x_n-x_0|}{t+|x_n-x_0|} = 0$

$$\Rightarrow \lim_{n\to\infty} |x_n - x_0| = 0$$

Now, for all t > 0 , $N_2(f(x_n) - f(x_0), t) = \frac{t}{t+k|f(x_n)-f(x_0)|}$

$$= \frac{t}{t+k\left|\frac{x_n^4}{1+x_n^2} - \frac{x_0^4}{1+x_0^2}\right|}$$

$$= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2) + k\left|x_n^4(1+x_0^2) - x_0^4(1+x_n^2)\right|}$$

$$= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+k\left|(x_n^2+x_0^2)(x_n^2+x_0^2)-x_n^2x_0^2(x_n^2+x_0^2)\right|}$$

$$\Rightarrow \lim_{n\to\infty} N_2\,(f(x_n) - f\,(x_0)\ ,\, t\ ) = 1.$$

$$\Rightarrow M_2(f(x_n) - f(x_0),\, t) = \frac{k\left|(x_n^2+x_n^2)(1+x_0^2)+x_n^2x_0^2(x_n^2+x_0^2)\right|}{t(1+x_n^2)(1+x_0^2)+k\left|(x_n^2+x_0^2)(x_n^2+x_0^2)-x_n^2x_0^2(x_n^2+x_0^2)\right|}$$

$$\lim_{n\to\infty} M_2\,(f(x_n) - f\,(x_0),\, t) = 0.$$

Thus, f is sequentially continuous on X . From the calculation of the example, it follows that f is not strongly IFC.

Theorem(1.3.13)[25] Let f be a mapping from the IFNLS (U, A) to (V, B) and D be a compact subset of U . If f IFC on U then f (D) is a compact subset of V.

Proof: Let $\{y_n\}_n$ be a sequence in f (D). Then for each n, $\exists\ x_n\ \varepsilon$, D such that f $(x_n) = y_n$. Since D is compact, there exists $\{x_{nk}\}_k$a subsequence of $\{x_n\}_n$ and $x_0\ \varepsilon$ D such that $x_{nk} \longrightarrow x_0$ in (U,A).

Since f is IFC at x_0 if for any given $\varepsilon > 0$, β (0, 1), $\exists\ \delta=\delta\,(\alpha, \varepsilon) > 0$, $\beta = \beta\,(\alpha, \varepsilon)$ (0, 1) such that for all x ε U,

$$N_U\,(x - x_0, \delta) > \beta \Rightarrow N_V\,(f\,(x) - f\,(x_0), \varepsilon) > \alpha$$

and

$$M_U\,(x - x_0, \delta) < 1 - \beta \Rightarrow M_V\,(f\,(x){-}f\,(x_0), \varepsilon) < 1{-}\alpha$$

Now, $x_{nk} \longrightarrow x_0$ in (U, A) implies that $\exists\ n_0\ \varepsilon$ N such that for all $k \geq n_0$

$$N_U\,(x_{nk} - x_0, \delta) > \beta \text{ and } M_U\,(x_{nk} - x_0, \delta\ ) < 1 - \beta$$

$$\Rightarrow N_V\,(f\,(x_{nk}) - f\,(x_0)\ , \varepsilon) > \alpha\ \varepsilon) < 1 - \alpha$$

$$\text{and } M_V\,(f\,(x_{nk}) - f\,(x_0)\ , \varepsilon) < 1 - \alpha$$

i. e. $N_V\,(y_{nk} - f\,(x_0), \varepsilon\ ) > \alpha$ and $M_V\,(y_{nk} - f\,(x_0), \varepsilon) < 1 - \alpha\ \forall\ k \geq n_0$

$\Rightarrow$f (D) is a compact subset of V.

Chapter 2

Fuzzy Anti – Normed Linear Space and Some Properties of B(X, Y)

Section(2-1) Fuzzy Anti – Normed Linear Space

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The idea of fuzzy norm was initiated by katsaras. Felbin defined a fuzzy norm on a linear space whose associated fuzzy metric is of kaleva-Seikkala type. Cheng and Mordeson introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type.

Bag and Samanta gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type. They also studied some properties of the fuzzy norm. Bag and Samanta discussed the notions of convergent sequence and Cauchy sequence in fuzzy normed linear space. They also made a comparative study of the fuzzy norms defined by Katsaras, Felbin, and Bag and Samanta.

After an introduction of fuzzy norms, we introduce a fuzzy anti-norm linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta and investigate their important properties. Then we shall introduce the notions of convergent sequence, Cauchy sequence in fuzzy anti-normed linear space. We also introduce the concept of compact subset and bounded subset in fuzzy anti-normed linear space.

Lastly, we have introduced the definition of intuitionistic fuzzy normed linear space.

In, Felbin introduced the concept of a fuzzy norm based on a kaleva- Seikkala type of fuzzy metric using the notion of fuzzy number. Let X be a vector space over R (set of real numbers). Let|| ||:X→R*(I) be a mapping and let the mappings L,U:

[0,1]×[0,1]→[0,1] be symmetric, nondecreasing in both arguments and satisfying L(0,0) = 0 and U (1,1) =1.Write $[\|x\|]_a = [\|x\|_a^1 R \|x\|_a^2]$ for $x \in X$,0<α≤1 and suppose for all x∈X , x≠0 there exists α ∈(0,1] independent of x <such that for all α ≤α_0,

(A) $\|x\|_a^2 < \infty$,

(B) $\inf \|x\|_a^1 > 0$.

The quadruple (X ,|| || ,L,U)is called a Felbin-fuzzy normed linear space and is a Felbin-fuzzy norm if:

(i) ||x|| = $\bar{0}$ if and only if x = $\underline{0}$ (the null vector),

(ii) ||rx|| =| r | ||x|| , x ∈ X , r ∈R

(iii) for all x , y ∈X,

(a) Whenever $s \geq \|x\|_1^1$, $t \geq \|y\|_1^1$, and $s + t \geq \|x + y\|_1^1$,

||x + y|| (s + t) ≥ L (||x|| (s), ||y|| (t)).

(b) Whenever $s \leq \|x\|_1^1$, $t \leq \|y\|_1^1$, and $s + t \leq \|x + y\|_1^1$,

||x + y|| (s + t) ≤ U (||x|| (s), ||y|| (t)).

Fuzzy Norm on A Linear Space(2.1.1)[11]

We devoted to a collection of basic definitions and results which will be needed.

Definition (2.1.2)[11] Let X be a linear space over a real field F (field of real/complex numbers). A fuzzy subset N of X ×R is called a fuzzy norm on X if the following conditions, are satisfied for all x, y ∈X.

(N1) For all $t \in R$ with $t \leq 0$, $N(x,t) = 0$,

(N2) For all $t \in R$ with $t > 0$, $N(x,t) = 1$ if and only if $x = 0$,

(N3) For all $t \in R$ with $t > 0$, $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$, $c \in F$,

(N4) For all $s, t \in R$, $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$,

(N5) $N(x,.)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x,y) = 1$.

Then N said to be a fuzzy norm on a linear space X and the pair (X ,N)is said to be a fuzzy normed linear space or in short FNLS.

The following condition of fuzzy norm N will be required later on.

(N6) $N(x,t) > 0, \forall\, t > 0$ implies $x = \underline{0}$.

Example(2.1.3)[11] Let (X,|| ||)be a normed linear space. Define

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & \text{when } t > 0,\ t \in R,\ x \in X, \\ 0, & \text{when } t \leq 0. \end{cases}$$

Then (U ,N) is an FNLS.

Example(2.1.4)[11] Let (X, || ||)be a normed linear space. Define

$$N(x,t) = \begin{cases} 0, & \text{if } t \leq \|x\|,\ t \in R,\ x \in X, \\ 1, & \text{if } t > \|x\|,\ t \in R,\ x \in X. \end{cases}$$

Then (U ,N) is an FNLS.

Theorem(2.1.5)[11](Let (X ,N)be a fuzzy normed linear space. Define $\|x\|_\alpha = \inf\{t : N(x,t) \geq \alpha\}$; $\alpha \in (0,1)$. Then $\{\| \|_\alpha : \alpha \in (0,1)\}$is an ascending family of norms on X. These norms are called $\alpha-$ norms on X corresponding to fuzzy norm on X .

Theorem(2.1.6)[11] Let$\{\| \|_\alpha : \alpha \in (0,1)\}$ be an ascending family of norms on linear space X . Define a function $N' : X \times R \to [0,1]$ as:

$$N'(x,t) = \begin{cases} \sup\{\alpha \in (0,1) : \|x\|_\alpha \leq t\}, & \text{when } (x,t) \neq 0, \\ 0, & \text{when } (x,t) = 0. \end{cases}$$

Then N' is a fuzzy norm on X.

If the index set (0,1) of the family of crisp norms$\{\|\ \|_\alpha : \alpha \in (0,1]\}$ of Theorem(2.1.5) is extended to (0,1] then a fuzzy norm N is generated, satisfying an additional property that N (x, ·)attains the value 1 at some finite value t.

Theorem(2.1.6)[11] Let $\{\|\ \|_\alpha : \alpha \in : (0,1)\}$be a descending family of norms on a linear space X. Now define a function

$$N' : X \times R \to [0,1] \text{ as}$$

$$N'(x,t) = \begin{cases} \sup\{\alpha \in (0,1] : \|x\|_\alpha \le t\}, & \text{when } (x,t) \ne 0, \\ 0, & \text{when } (x,t) = 0. \end{cases}$$

Then

(a) N' is a fuzzy norm on X .

(b) For each $x \in X$, $\exists t = t(x) > 0$ such that $N'(x , s) = 1, \forall s \ge t$.

Definition (2.1.8)[11] Let (X ,N) be a fuzzy normed linear space. Let$\{n_x\}$ be a sequence in X . Then $\{n_x\}$is said to be convergent if $\exists x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1, \ \forall t > 0$.

Definition(2.1.9)[11] Let (X ,N) be a fuzzy normed linear space. Let $\{n_x\}$be a sequence in X . Then $\{n_x\}$is said to be a Cauchy sequence if $\lim_{n \to \infty} N(x_{n+p} - x_n, t) = 1$, $\forall t > 0$ and $\{n_x\}$, p = 1,2,3,...

Definition(2.1.10)[11] A subset of a fuzzy normed linear space (U,N^*) is said to be bounded if and only if $\exists t > 0$ and $0 < r < 1$such that $N^*(x, t) > 1 - r, \forall x \in A$.

Definition(2.1.11)[11] A subset A of a fuzzy normed linear space (U,N^*) is said to be compact if any sequence$\{n_x\}$in A has a subsequence converging to an element of A.

Fuzzy Anti-norm on A Linear Space (2.1.12)[11]

We introduce the notion of fuzzy anti-normed linear space and investigate their important properties.

Definition(2.1.13)[11] Let U be a linear space over a real field F. A fuzzy subset N^* of X × R such that for all x ,u∈U and c ∈F:

(N^*1) For all t ∈R with $t \leq 0$, N^* (x, t) =1;

(N^*2) For all t ∈R with $t > 0$, N^* (x, t) = 0 if and only if x = 0;

(N^*3) For all t ∈R with $t > 0$, N^* (cx, t) =N^* (x, t/|c|) if $c \neq 0$, c ∈F;

(N^*4) For all s ,t ∈R , N^* (x + u, s + t) ≤ max{N^* (x, s) =N^* (u, t)};

(N^*5) N^*(x, t) is a non-increasing function of t ∈ R and $\lim_{t \to \infty} N^*$ (x, t)= 0. Then N is said to be a fuzzy anti-norm on a linear space U and the pair (U,N^*) is called a fuzzy anti-normed linear space or in short Fa-NLS.

The following condition of fuzzy norm N. will be required later on.

(N^*6) For all t ∈R with $t > 0$, N^*(x, t) <1 implies x = 0 .

Example(2.1.14)[11] Let (U, || ||) be a normed linear space. Define

$$N(x,t)=\begin{cases}\dfrac{\|x\|}{t+\|x\|}, & \text{when } t>0,\ t\in R,\ x\in U,\\ 0, & \text{when } t\leq 0.\end{cases}$$

Then (U, N^*) is an Fa-NLS.

Proof: Now we have to show that N^*(x, t)is afuzzy anti-norm in U.

(N^*1) For all t ∈R . If $t \leq 0$, we have by definition N^*(x, t).

(N^*2) For all t ∈R with $t > 0$, $N^*(x, t) = 0 = \Leftrightarrow \dfrac{\|x\|}{t+\|x\|} = 0 \Leftrightarrow x = \underline{0}$.

(N*3) For all t ∈ R with t > 0, and <c ($\neq$ 0)∈ F (field of real/complex numbers), we get

$$N^*(cx,t)=\frac{\|cx\|}{t+\|cx\|}=\frac{|c|\|x\|}{t+|c|\|x\|}=\frac{\|x\|}{\frac{t}{|c|}+\|x\|}=N^*\left(c,\frac{t}{|c|}\right).$$

(N^*4) For all s ,t ∈ R and x ,u ∈ U . We have to show that N^*(x+u, s,t) ≤ max{N^*(x, s), N^*(u, t)}. If (a) s +t < 0, (b) s = t = 0,

(c) s +t > 0; s > 0, t < 0; s < 0, t > 0, then in these cases the relation is obvious. If (d) s > 0,t > 0,s +t > 0. Then, assume that

$$N^{*}(x,s)\leq N^{*}(u,t)=\frac{\|x\|}{s+\|x\|}\leq\frac{\|u\|}{t+\|u\|}$$
$$=\|x\|(t+\|u\|)\leq\|u\|(s+\|x\|)\Rightarrow t\|x\|\leq s\|u\|\ldots \quad (1)$$

Now

$$\frac{\|x+u\|}{s+t+\|x+u\|}-\frac{\|u\|}{t+\|u\|}\leq\frac{\|x\|+\|u\|}{s+t+\|x+u\|}-\frac{\|u\|}{t+\|u\|}=\frac{t\|x\|-s\|u\|}{(s+t+\|x\|+\|u\|)(t\|u\|)}.$$

By (1),

$$\frac{\|x+u\|}{s+t+\|x\|+\|u\|}\leq\frac{\|u\|}{t+\|u\|}\quad \text{similarly}\quad \frac{\|x\|+\|u\|}{s+t+\|x\|+\|u\|}\leq\frac{\|x\|}{t+\|x\|}.$$

$N^{*}(x+u, s,t) \leq \max\{N^{*}(x, s), N^{*}(u, t)\}$.

($N^{*}5$) If $t_1 < t_2 \leq 0$, then we have $N^{*}(x, t_1)$, $N^{*}(x, t_2) = 0$. If $t_1< t_2\leq 0$ then

$$\frac{\|x\|}{t_1+\|x\|}\leq\frac{\|x\|}{t_2+\|x\|}=\frac{\|x\|(t_2-t_1)}{(t_1+\|x\|)(t_2+\|x\|)}>0\Rightarrow N^{*}(x,t_1)\geq N^{*}(x,t_2).$$

Thus $N^{*}(x ,\cdot)$ is a non-increasing of R. Again if $x \neq \underline{0}$ then

$$\lim_{t\to\infty} N^{*}(x,t)=\lim_{t\to\infty}\frac{\|x\|}{t+\|x\|}=0.$$

IF $x = \underline{0}$ then $\qquad \lim_{t\to\infty} N^{*}(x,t)=N^{*}(\underline{0},t)=0.$

Thus $\lim_{t\to\infty} N^{*}(x,t)= 1$, $\forall x \in U$. Hence (U ,N^{*}) is an Fa-NLS.

Example(2.1.15)[11]Let (U, || ||) be a normed linear space. Define N^{*}: X × R → [0,1] by

$$N^{*}(x,t)=\begin{cases}0, & \text{if } \ t>\|x\|,\ t\in R,\ \ x\in U,\\ 1, & \text{if } \ t<\|x\|,\ t\in R,\ \ x\in U.\end{cases}$$

Then (U ,N^{*}) is an Fa-NLS.

Proof: It can be easy verified that (U ,N^{*}) is Fa-NLS.

Remark(2.1.16)[11] N^* is a fuzzy anti-norm on U iff $1- N^*$ is a fuzzy norm on U .

Lemma(2.1.17)[11] Let(U,N^*)be aFa-NLS. Then $N^*(x-y, t)= N^*(y-x,t)$ for all x , y in U and t $\in(0,\infty)$.

Proof: For x , y in U and t $\in(0,\infty)$,N* (x − y ,)= N^* (−(y – x) ,t) = N^* (y − x ,t / |−1|) N^* (y − x ,t) .

Definition(2.1.18)[11] Let N^* be afuzzy anti-norm U satisfying (N^*6). Define $\|x\|_\alpha^* \inf\{t>0: N^*(x,t)<\alpha, \alpha\in(0,1]\}$.

Lemma(2.1.19)[11] Let (U ,N^*) be a Fa-NLS. For each $\alpha \in(0,1]$ and x $\in$U. Then we have

(i) $\|x\|_{\alpha_1}^* \ge \|x\|_{\alpha_2}^*$, for $0 < \alpha_1< \alpha_2 \le 1$.

(ii) $\|cx\|_\alpha^* = |c|\|x\|_\alpha^*$, for any scalar c.

(iii) $\|x+y\|_\alpha^* \le \|x\|_\alpha^* + \|x\|_\alpha^*$.

Proof: (i) For $0 < \alpha_1< \alpha_2 \le 1$, we note inf{t > 0: N^* (x ,t)< α_2}

Inf {t > 0: N^* (x ,t)< α_2}$\Rightarrow$ $\|x\|_{\alpha_1}^* \ge \|x\|_{\alpha_2}^*$.

(ii) For any scalar c and $\forall\alpha \in(0,1]$,

$\|x\|_\alpha^* + \|x\|_\alpha^*$= inf{t > 0: $N^*(x ,t)< \alpha$}+ inf{s > 0: N^* (y ,s)< α}

$\ge$ inf{t + s > 0: $N^*(x ,s)< \alpha$,$N^*(y ,s)< \alpha$}=$\|x+y\|_\alpha^*$.

Theorem(2.1.20)[11] Let (U ,N^*) be a Fa-NLS. Then $\{\|\ \|_\alpha^*:\alpha \in(0,1]\}$ is a decreasing family of norms on U.

Proof: By Lemma(2.1.19) it can be easy verified that.

Theorem(2.1.21)[11] Let $\{\|\ \|_\alpha^*:\alpha \in(0,1]\}$be an decreasing family of norms on linear space U. Now define a function

N^*_1: U×R→[0,1]as

$$N_1^*(x,t) = \begin{cases} \inf \{\alpha \in (0,1] : \|x\|_\alpha^* \le t\}, & \text{when } (x,t) \ne 0, \\ 1, & \text{when } (x,t) \ne 0. \end{cases}$$

Then

(a) $N^*{}_1$ is a fuzzy anti-norm on U .

(b) For each x ∈U , ∃r = r (x) > 0 such that $N_1^*(x,t)$ =1.

Proof: Now we have to show that N_1^* is a fuzzy anti-norm on X .

(N^*1) (a) ∀t ∈R with t < 0, $\{\alpha \in (0,1] : \|x\|_\alpha^* \le t\}$= ϕ ∀ ∈ we have $N_1^*(x,t)$ = 1.

(b) For t = 0 and x ≠ $\underline{0}$, $\{\alpha \in (0,1] : \|x\|_\alpha^* \le t\}$= ϕ ∀ ∈ we have $N_1^*(x,t)$ = 1. $\alpha \in (0,1] : \|x\|_\alpha^* \le t$

(c) For t = 0 and x = $\underline{0}$ then from the definition $N_1^*(x,t)$ = 1.

Thus ∀t ∈R with $N_1^*(x,t)$ =1,∀x ∈U .

(N^*2) ∀t ∈R with t > 0, $N_1^*(x,t)$ = 0. Choose any ε ∈(0,1). Then for any $t > 0, \alpha_1 \in (\alpha,1]$such that $\|x\|_{\alpha_1}^* \le t$, and hence $\|x\|_\varepsilon^* \le t$, since t > 0 is arbitrary, this implies that $\|x\|_\varepsilon^*$ = $\underline{0}$.

If x = $\underline{0}$ then for t > 0, $N_1^*(\underline{0},t)$= inf $\{\alpha \in (0,1] : \|\underline{0}\|_\alpha^* \le t\}$= 0. Thus for all t ∈R with t > 0, $N_1^*(x,t)$ = 0 if and only if x = $\underline{0}$.

(N^*3) For all t ∈R with t > 0, and c (≠ 0)∈F, we have

$$N_1^*(cx,t) = \inf\{\alpha \in (0,1] : \|cx\|_\alpha^* \le t\} = \inf\{\alpha \in (0,1] : |c|\|x\|_\alpha^* \le t\}$$

$$= \inf\left\{\alpha \in (0,1] : \|cx\|_\alpha^* \le \frac{t}{|c|}\right\} = N_1^*\left(x, \frac{t}{|c|}\right), \ \forall x \in U.$$

(N^*4) We have to show that ∀s ,t ∈R and

∀x ,u ∈U , $N_1^*(x+u, s+t) \le \max\{N_1^*(x,s), N_1^*(u,t)\}$.

Suppose that ∀s,t∈R and ∀x,u∈U, $N_1^*(x+u,s+t)$>max $\{N_1^*(x,s), N_1^*(u,t)\}$

Choose k such that $N_1^*(x+u,s+t)$ > k > max $\{N_1^*(x,s), N_1^*(u,t)\}$.

Now
$$N_1^*(x+u,s+t) > k \Rightarrow \inf\{\alpha \in (0,1] : \|x+u\|_\alpha^* \le s+t\}$$
$$> k \Rightarrow \|x+u\|_\alpha^* \le s+t \Rightarrow \|x\|_k^* \|u\|_k^* > s+t.$$

Again

$$k \Rightarrow \max\{N_1^*(x,s), N_1^*(x,s)\} \Rightarrow k > N_1^*(x,s) \text{and} \, k > N_1^*(u,t) \|x\|_k^* \le s \,\text{and}\, \|u\|_k^* \le t$$
$$\Rightarrow \|x\|_k^* \|ku\|_k^* > s+t.$$

Thus $s+t < \|x\|_k^* + \|u\|_k^* \le s+t$, a condition.

Hence $N_1^*(x+u,s+t) \le \max\{N_1^*(x,s), N_1^*(x,s)\}$.

Definition (2.1.22)[11] Let (U ,N^*) be a Fa-NLS. A sequence $\{x_n\}$ in U is said to be convergent to x ∈U if given t > 0,0 < r <1there exists an integer n ∈N such that $N_1^*(x_n - x, t)$ < r, for all n ≥ n_0.

Example(2.1.23)[11] Let (X, || ||)be an normed linear space and N^* :X ×R→ [0,1]. Define

$$N^*(x,t) = \begin{cases} \dfrac{\|x\|}{t+\|x\|}, & \text{when } t > 0, \ t \in R, \\ 1, & \text{when } t \le 0. \end{cases}$$

Then (X ,N^*) is a Fa-NLS (see Example(2.1.16). Let $\{x_n\}$be a sequence in X , then

a) $\{x_n\}$ is a Cauchy sequence in (X, || ||) if and only if $\{x_n\}$is an Cauchy sequence in (X ,N^*).

b)$\{x_n\}$ is a convergent sequence in (X, || ||) and only if $\{x_n\}$is an convergent sequence in (X ,N^*).

Proof: a) Let $\{x_n\}$be a Cauchy sequence in (X, || ||). Then

$$\Leftrightarrow \lim_{n\to\infty} \|x_n - x_{n+p}\| = 0, \quad \text{for all } p = 1,2,3,...$$

$$\Leftrightarrow \lim_{n\to\infty} N^*(x_n - x_{n+p}) = \lim_{n\to\infty} \frac{\|x_n - x_{n+p}\|}{t + \|x_n - x_{n+p}\|} = 0, \quad \text{for all} \quad t > 0.$$

$$\Leftrightarrow \lim_{n\to\infty} N^*(x_n - x_{n+p}) = 0.$$

$\Leftrightarrow \{x_n\}$ is an Cauchy sequence in (X ,N^*).

b) $\{x_n\}$is a convergent sequence in (X, || ||). Then

$$\Leftrightarrow \lim_{n\to\infty} \|x_n - x\| = 0.$$

$$\Leftrightarrow \lim_{n\to\infty} N^*(x_n - x) = \lim_{n\to\infty} \frac{\|x_n - x\|}{t + \|x_n - x\|} = 0, \quad \text{for all} \quad t > 0.$$

$$\Leftrightarrow \lim_{n\to\infty} N^*(x_n - x) = 0.$$

$\Leftrightarrow \{x_n\}$ is an convergent sequence in (X ,N^*).

Definition(2.1.24)[11] Let (X ,N^*) be a Fa-NLS. A subset B of U is said to be closed if for any sequence $\{x_n\}$ in B converges to x ∈B , that is, $\lim_{n\to\infty} N^*(x_n - x) = 0$, ∀t >0 implies that x ∈ B.

Definition(2.1.25)[11] Let (X ,N^*) be a Fa-NLS. A subset W of U is said to be the closure of B ⊂W if for any w ∈W , there exists a sequence $\{x_n\}$in B such that $\lim_{n\to\infty} N^*(x_n - x) = 0$, ∀t ∈ R^+, we denote the set W by B.

Definition(2.1.26)[11] A subset A of a Fa-NLS is said to be bounded if and only if ∃t>0 and 0<r<1such that $N_1^*(x,t) < r$, ∀x ∈A.

Definition(2.1.27)[11] Let (X ,N^*) be a Fa-NLS. A subset A of a Fa-NLS is said to be compact if any sequence$\{x_n\}$in A has a subsequence converging to an element of A.

Intuitition Fuzzy Noem(2.1.28)[11]

We redefine the notion of fuzzy normed linear space using t-norm and fuzzy anti-normedlinear space using t-conorm then we introduce the definition of intuitionistic fuzzy norm over a linear space.

Definition(2.1.29)[11] A binary operation $*$: $[0,1]\times[0,1]\rightarrow[0,1]$is continuous t-norm if $*$ is it satisfies the following conditions:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a *1 = a$ for all $a\in[0,1]$;

(d) $a *b \le c *d$ whenever $a \le c$, and $b \le d$, and $a,b,c,d \in[0,1]$.

Examples of continuous t-norm are $a *b = ab$,$a *b = \min\{a,b\}$ and $a *b = \max\{a +b -1,0\}$.

Definition(2.1.30)[11] Let X be a linear space over a real field F (field of real/complex numbers). A fuzzy subset N of $X \times R$ (set of real numbers) is called a fuzzy norm on X if the following conditions, are satisfied for all $x , y \in X$:

(N1) For all $t \in R$ with $t \le 0$,$N (x ,t) = 0$,

(N2) For all $t \in R$, $N (x ,t) =1$if and only if $x = 0$,

(N3) For all $t \in R$ with $t > 0$,$N (cx ,t) = N (x ,t / c)$ if $c \ne 0$,$c \in F$,

(N4) For all $s ,t \in R$, $N (x + y ,s +t) \ge N (x ,s)*N (y ,t)$,

(N5) $N (x ,\cdot)$s a non-decreasing function of R and $\lim_{t\rightarrow\infty} N (x,t) =1$

Then N is said to be a fuzzy $*-$ norm on a linear space X.

Definition(2.1.31)[11]A binary operation $\Diamond$: $[0,1]\times[0,1]\rightarrow[0,1]$ is continuous t-conorm if $\Diamond$ is it satisfies the following conditions:

(a) $\Diamond$ is commutative and associative;

(b) $\Diamond$ is continuous;

(c) $a\Diamond 0 = a$ for all $a\in[0,1]$;

(d) $a\lozenge b \le c\lozenge d$ whenever $a \le c$, and $b \le d$, and $a,b,c,d \in[0,1]$.

Examples of continuous t-conorm are $a\lozenge b = \min\{a +b,1\}$, $a\lozenge b = \max\{a,b\}$ and $a\lozenge b = a +b -ab$.

Definition(2.1.32)[11] Let U be a linear space over a real field F . A fuzzy subset M of X ×R such that for all $x ,u \in U$ and $c \in F$:

(N^*1) For all $t \in R$ with $t \le 0$, $M (x ,t) =1$;

(N^*2) For all $t \in R$ with $t > 0$, $M (x ,t) = 0$ if and only if $x = 0$;

(N^*3) For all $t \in R$ with $t > 0$, $M (cx ,t) = M (x ,t / \ )$ if $c \neq 0, c \in F$;

(N^*4) For all $s ,t \in R$, $M^*(x +u, s +t)\le M(x , s)\lozenge N(u ,t)$;

(N^*5) $M (x ,t)$ is a non-increasing function of $t \in R$ and $\lim_{t\to\infty}M(x, t) = 0$.

Then M is said to be a fuzzy $\lozenge -$ antinorm on a linear space U.

Definition(2.1.33)[11] Let*be a continuous t-norm, $\lozenge$ be a continuous t-conorm and V be a linear space over the field F (= R or C).An intuitionistic fuzzy norm on V is an object of the form A $\{((x ,t),N (x ,t),M (x ,t)) : (x ,t)\in V\times R^+ \}$, where N , M are fuzzy sets on $V \times R^+$, N denotes the degree of membership and M denotes the degree of non membership $(x ,t)\in V \times R^+$ satisfying the following conditions:

(i) N is a fuzzy $*-$ norm on a linear space V .

(ii) M is a fuzzy $\lozenge -$ antinorm on a linear space V .

(iii) $N (x ,t) + M (x ,t) \in 1, \forall (x ,t)\in V \times R^+$.

Definition(2.1.34)[11] If A is an intuitionistic fuzzy norm on V (a linear space over the field F (= R or C) then (V ,A) is called an intuitionistic fuzzy normed linear space or in short IFNLS.

Though there are the concepts of fuzzy inner product space but the concept of fuzzy norm could not be induced by these concepts of fuzzy inner product. So, one can develop the concept of fuzzy inner product which can induce the concept of fuzzy norm. Also, one can develop the concept of anti fuzzy inner product which can induce the concept of anti fuzzy norm.

Section (2-2) On Felbin's-Type Fuzzy Normed LINEAR Spaces and Fuzzy Bounded Operators

An idea of fuzzy norm on a linear space first introduced by Katsaras. Felbin defined a fuzzy norm (the induced fuzzy metric of which is of Kaleva and Seikkala's type), by assigning a non-negative fuzzy real number to each element of a linear space. A further developments along this line of inquiry took place in 1994, when Cheng and Mordeson evolved the definition of a further type of fuzzy norm having a corresponding metric of the Kramosil and Michalek's type. Bag and Samanta considered a fuzzy norm slightly different from the one defined by Cheng and Mordeson and for which a suitable decomposition theorem was proved. Based on this theorem it has been possible to establish four fundamental theorems of functional analysis, the Hahn-Banach theorem, the open mapping theorem, the closed graph theorem and the uniform boundedness principle. Also best approximation in this space is studied. Fuzzy bounded linear operators in Felbin's-type fuzzy normed spaces were introduced by M. Itoh and M. Cho. They introduced a fuzzy norm for fuzzy bounded operators. Felbin introduced an idea of fuzzy bounded operators and defined a fuzzy norm for such an operator which was erroneous as shown in **Example(2.2.1)[17]** Xiao and Zhu, studied various properties of Felbin's-type fuzzy normed linear spaces and a new definition for norm of bounded operators was discussed.

A different definition of a fuzzy bounded linear operator and a ”fuzzy norm” for such an operator was introduced by Bag and Samanta. The dual of a fuzzy normed space and a Hahn-Banach’s theorem for fuzzy strongly bounded linear functional were established.

A comparative study among several types of fuzzy norms on a linear space defined by various authors, has been made by Bag and Samanta. They classified these norms into two types, one of which is Katsaras’s type, and the other is Felbin’s type.

Preliminaries(2.2.2)[17]

According to Mizumoto and Tanaka, a fuzzy number is a mapping $x: R \longrightarrow [0, 1]$ over the set R of all reals.

x is called convex if $x(t) \geq \min(x(s), x(r))$ where $s \leq t \leq r$.

If there exists a $t_0 \in R$ such that $x(t_0)=1$, then x is called normal. For $0<\alpha\leq 1$, α-level set of an upper semicontinuous convex normal fuzzy set x of R(denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible.

When $a_\alpha=-\infty$, for instance, then $[a_\alpha,b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similar is the case when $b_\alpha = +\infty$. x is called non-negative if for all $t < 0$, $x(t) = 0$. Kaleva and Seikkala (Felbin) denoted the set of all convex, normal, upper semi-continuous fuzzy real numbers by E(R(I)) and the set of all non-negative, convex, normal, upper semi-continuous fuzzy real numbers by $G(R^*(I))$.

As α-level sets of a convex fuzzy number is an interval, there is a debate in the nomenclature of fuzzy numbers/fuzzy real numbers. Dubois and Prade suggested to call this as fuzzy interval. They developed a different notion of a fuzzy real number by considering it as a fuzzy element of the real line, each α-cut of this number is an interval real numbers. From now on ”fuzzy real numbers” are renamed as ”fuzzy intervals”. While refereing to previous results involving fuzzy real number, the term fuzzy interval is written within brackets after fuzzy real number to avoid any confusion; otherwise the new nomenclature i.e. fuzzy interval is used.

We consider the concept of fuzzy real numbers (fuzzy intervals) in the sense of Xiao and Zhu which is defined below:

A mapping $\eta: R \longrightarrow [0, 1]$, whose α-level set is denoted by $[\eta]_\alpha := \{t: \eta(t) \geq \alpha\}$, is called a fuzzy real number (or fuzzy interval) if it satisfies two axioms:

(N_1) There exists $t_0 \in R$ such that $\eta(t_0) = 1$.

(N_2) For each $\alpha \in (0, 1]$; $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$, where $-\infty < \eta_\alpha^1 \leq \eta_\alpha^2 < \infty$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by F. For each $\bar{r}r \in R$,

let $\bar{r} \in F$ be defined by $\bar{r}(t) = 1$, if $t = r$ and $\bar{r}(t) = 0$, if $t \neq r$, so $\bar{r}$ is a fuzzy interval and R can be embedded in F.

Let $\eta \in F$, η is called positive fuzzy real number if for all $t < 0$, $\eta(t) = 0$. The set of all positive fuzzy real numbers (fuzzy interval) is denoted by F^+.

A partial order $\preceq$ in F is defined as follows, $\eta \preceq \delta$ if and only if for all $\alpha \in (0, 1]$, $\eta_\alpha^1 \leq \delta_\alpha^1$ and $\eta_\alpha^2 \leq \delta_\alpha^2$ where, $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$ and $[\delta]_\alpha = [\delta_\alpha^1, \delta_\alpha^2]$. The strict inequality in F is defined by $\eta \prec\_\delta$ if and only if for all $\alpha \in (0, 1]$, $\eta_\alpha^1 < \delta_\alpha^1$ and $\eta_\alpha^1 < \delta_\alpha^2$.

Kaleva and Seikkala, proved a sufficient condition for a family of intervals to represent the α-level sets of a fuzzy real number. In fact, let $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$

be a given family of nonempty intervals. If:

(i) for all $0 < \alpha_1 \leq \alpha_2$, $[a_{\alpha_1}, b_{\alpha_1}] \supseteq [a_{\alpha_2}, b_{\alpha_2}]$.

(ii) $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$, whenever $\{\alpha_k\}$ is an increasing sequence in (0, 1] converging to α, then the family $[a_\alpha, b_\alpha]$ represents the α-level sets of a fuzzy real number (fuzzy interval). Conversely, if $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy number then the condition (i) and (ii) are satisfied.

According to Mizumoto and Tanaka , the arithmetic operations ⊕, Θ, ʘ on F × F are defined by

$$(x \oplus y)(t) = \sup_{s\in R} \min\{x(s), y(t-s)\},\ t \in R,$$

$$(x\ \Theta\ y)(t) = \sup_{s\in R} \min\{x(s), y(s-t)\},\ t \in R,$$

$$(x \odot y)(t) = \sup_{0\neq s\in R} \min\{x(s), y(\tfrac{t}{s})\},\ t \in R.$$

We also consider an operation Θ on $\eta \in F$ and $\delta(\succ 0)$ 2 F^+ as follows $(\eta\ \Theta\ \delta)(t) = \sup_{s\in R} \min\{\eta(st), \delta(s)\}$, $t \in R$.

We know that, for $\eta, \delta \in F$, if$[\eta]_\alpha=[\eta^1_\alpha, \eta^2_\alpha]$,$[\delta_\alpha]=[\delta^1_\alpha, \delta^2_\alpha]$,$\alpha\in(0, 1]$, Then

$$[\eta \oplus \delta]_\alpha = [\eta^1_\alpha + \delta^1_\alpha, \eta^2_\alpha + \delta^2_\alpha],$$

$$[\eta \Theta \delta]_\alpha = [\eta^1_\alpha - \delta^1_\alpha, \eta^2_\alpha - \delta^2_\alpha],$$

furthermore if $\eta, \delta \in F+$, then$[\eta \odot \delta]_\alpha=[\eta^1_\alpha . \delta^1_\alpha, \eta^2_\alpha . \delta^2_\alpha]$, and when $\delta \succ \bar{0}$,$[\bar{1} \Theta\ \delta]_\alpha = [\frac{1}{\delta^2_\alpha}, \frac{1}{\delta^1_\alpha}]$. Now one can see that for $\delta \succ 0$ and $\eta \in F_+$,

$[\eta\ \Theta\ \delta]_\alpha = [\frac{\eta^2_\alpha}{\delta^2_\alpha}, \frac{\eta^1_\alpha}{\delta^1_\alpha}]$.

A definition of fuzzy norm on a linear space was introduced by Felbin. Bag and Samanta, changed slightly this definition to define a fuzzy norm on a linear space as given below.

Definition(2.2.3)[17] Let X be a linear space over R. Suppose $\|\cdot\|: X \longrightarrow F^{+}$ is a mapping satisfying

(i) $\| x \| = \bar{0}$ if and only if $x = 0$,

(ii) $\| rx \| = | r | \| x \|$, $x \in X$, $r \in R$,

(iii) for all $x, y \in X$, $\| x + y \| \preceq \| x \| \oplus \| y \|$

And (A'): $x \neq 0 \Rightarrow \| x \| (t) = 0$, $8\forall \leq 0$.

$(X, \|\cdot\|)$ is called a fuzzy normed linear space and $\|\cdot\|$is called a fuzzy norm on X.

We use the previous definition of fuzzy norm. We note that

(i) condition(A') in Definition (2.2.3) is equivalent to the condition

(A''): For all $x(\neq 0) \in X$ and each $\alpha \in (0, 1]$, $\|x\|_{\alpha}^{1} > 0$,

where $[\|x\|_{\alpha}]_{\alpha} = [\|x\|_{\alpha}^{1}]_{\alpha}, [\|x\|_{\alpha}^{2}]_{\alpha}$and

(ii) $\|x\|_{\alpha}^{i}$, i = 1, 2, are crisp norms on X.

Let $(X, \| . \|)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if and only if for each $\alpha \in (0, 1]$, $\lim_{n\to\infty} \|x_n - x\|_{\alpha}^{2} = 0$. In this case we write $\lim_{n\to\infty} x_n = x$. Also a sequence $\{x_n\}$ is called a Cauchy sequence if for each $\alpha \in (0, 1]$, $\lim_{n,m\to\infty} \|x_n - x_m\|_{\alpha}^{2} = 0$. A fuzzy normed linear space $(X, \| . |)$ is said to be complete if every Cauchy sequence in X converges in X.

Proposition(2.2.4)[17]Let $\{[a_{\alpha}, b_{\alpha}]: \alpha \in (0,1]\}$, be a family of nested bounded closed intervals and $\eta: R \longrightarrow [0, 1]$ be a function defined by $\eta(t) = \vee\{\alpha \in (0, 1] : t \in [a_{\alpha}, b_{\alpha}]\}$. Then η is a fuzzy real number (fuzzy interval). The fuzzy real number (fuzzy interval) η which is constructed in Proposition(2.2.4) is called the fuzzy real number (fuzzy interval) generated by the family of nested bounded closed intervals $\{[a_\alpha, b_\alpha]:\alpha\in(0, 1]\}$. In this case, for

$\beta< \alpha,[\eta]_\alpha= [\eta^1_\alpha, \eta^2_\alpha]\subseteq[a_\beta, b_\beta]$. Now if $\beta> \alpha$, then $[a_\beta, b_\beta] \subseteq [\eta^1_\alpha, \eta^2_\alpha]$. In fact for $t \in [a_\beta, b_\beta]$, $\eta(t) = \vee\{\gamma \in (0, 1]: t \in [a_\gamma, b_\gamma]\} \geq \beta > \alpha$, which implies that, $t\in [\eta^1_\alpha, \eta^2_\alpha]$.

Proposition(2.2.5)[17] If η_i, $i = 1, 2$, are the fuzzy real numbers (fuzzy intervals) generated by the family of nested bounded closed intervals $\{[a^i_\alpha, b^i_\alpha]: \alpha \in (0, 1]\}$, $i = 1, 2$, and for each $\alpha\in (0, 1]$, $a^1_\alpha \leq a^2_\alpha, b^1_\alpha \leq b^2_\alpha$, then $\eta_1 \preceq \eta_2$.

We know that if η is a fuzzy real number (fuzzy interval) with $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$, and η^* is the fuzzy number (fuzzy interval) generated by the family of nested bounded closed intervals $[\eta^1_\alpha, \eta^2_\alpha]$, $0 < \alpha \leq 1$, then $\eta = \eta^*$.

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be two fuzzy normed linear spaces. A function $T: X \longrightarrow Y$ is said to be weakly fuzzy continuous at $x_0 \in X$ if for a given $\varepsilon > 0$, $\exists\delta \in F^+$, $\delta \succ 0$, such that

$$\|Tx - Tx_0\|^{\sim 1}_\alpha < \varepsilon \quad \text{whenever} \quad \|x - x_0\|^2_\alpha < \delta^2_\alpha,$$

$$\|Tx - Tx_0\|^{\sim 2}_\alpha < \varepsilon \quad \text{whenever} \quad \|x - x_0\|^1_\alpha < \delta^1_\alpha,$$

where for $\varepsilon \in (0, 1]$, $[\delta]_\alpha = [\delta^1_\alpha, \delta^2_\alpha]$.

Also a linear mapping $T : X \longrightarrow Y$ is called weakly fuzzy bounded if there exists a fuzzy interval $\eta \in F^+$, $\eta \succ 0$, such that for each $x(\neq 0) \in X$, $\|Tx\|^\sim \ominus \|x\| \succeq \eta$.

In this case the set of all weakly fuzzy bounded operators defined from X to Y is denoted by B(X, Y). In the sequel, we simply apply ”fuzzy continuous” ϴand ”fuzzy bounded” instead of ”weakly fuzzy continuous” and ”weakly fuzzy bounded”, respectively.

We know that a linear mapping T : X ⟶ Y is fuzzy continuous if and only if it is fuzzy bounded. Also the set B(X, Y) is a linear space with respect to usual operations.

The following result of Bag and Samanta is essential. Let (X, || . ||) and (Y, $||.k||^{\sim}$) be two fuzzy normed linear spaces and T ∈ B(X, Y). By definition ∃η 2 F^{+}, η ≻ 0, such that for all x(≠ 0) ∈ X,

$$\|Tx\|^{\sim} \ominus \|x\| \preceq \eta .$$

If $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2], 0 < \alpha \le 1$, we get

$$\|Tx\|_\alpha^{\sim 1} \le \eta_\alpha^1 \cdot \|x\|_\alpha^2 \quad \text{and} \quad \|Tx\|_\alpha^{\sim 2} \le \eta_\alpha^2 \cdot \|x\|_\alpha^1 .$$

Define

$$\|Tx\|_\alpha^{*1} = \sup_{0 \ne x \in X} \left\{ \frac{\|Tx\|_\alpha^{\sim 1}}{\|x\|_\alpha^2} \right\} \le \left(\eta_\alpha^1\right),$$

and

$$\|Tx\|_\alpha^{*2} = \sup_{0 \ne x \in X} \left\{ \frac{\|Tx\|_\alpha^{\sim 2}}{\|x\|_\alpha^1} \right\} \le \left(\eta_\alpha^2\right).$$

Then $\{\|\,.\,\|_\alpha^{*2} : \alpha \in (0, 1]\}$ and $\{\|\,.\,\|_\alpha^{*1} : \alpha \in (0, 1]\}$ are descending and ascending family of norms, respectively. Thus

$\{[\|Tx\|_\alpha^{*1}, \|Tx\|_\alpha^{*2}] : \alpha \in (0, 1]\}$ is a family of nested bounded closed intervals in R. Define the function $\|T\|^*$: R ⟶ [0, 1] by

$$\|T\|^*(t) = \vee\left\{\alpha \in (0,1] : t \in \left[\|Tx\|_\alpha^{*1}, \|Tx\|_\alpha^{*2}\right]\right\}.$$

$\|T\|^*$ is called the fuzzy norm of T. (B(X, Y), $||.||^*$) is a fuzzy normed space.

Section(2-3) Some Properties of B(X, Y)

We present some properties of the space (B(X, Y), $\|.\|^*$). Then using the recent results of Bag and Samanta, some consequences of fuzzy linear spaces analogous to the ordinary normed spaces are established. Despite our expectation, the fuzzy version of some well-known theorems in functional analysis, such as uniform boundedness principle, inverse mapping theorem and the Banach-Stienhaus's theorem is not valid in this fuzzy setting. These will be shown with some counterexamples. Next, finite dimensional normed spaces are considered. The concept of equivalent norms is defined and it is proved that every two fuzzy norms on a finite dimensional vector space are equivalent.

First we prove a memorable result for B(X, Y), which has a famous analogous in functional analysis. A similar result on B(X,C) is proved.

Theorem(2.3.1)[17] Let (X, $\|.\|$) be a fuzzy normed space and (Y, $\|.\|^{\sim}$) be a complete fuzzy normed space, then (B(X, Y), $\|\,.\,\|^*$) is a complete fuzzy normed linear space.

Proof Let $\{T_n\}$ be a Cauchy sequence in (B(X, Y), $\|\,.\,\|^*$). So for all $\alpha \in (0, 1]$,

$$\lim_{n,m\to\infty}\|T_n - T_m\|_\alpha^{*1} = \lim_{n,m\to\infty}\|T_n - T_m\|_\alpha^{*2} = 0.$$

From $\lim_{n,m\to\infty}\|T_n - T_m\|_\alpha^{*1} = 0$ we have

$$\lim_{n,m\to\infty}\|T_n - T_m\|_\alpha^{*1} \sup_{0\neq x\in X}\frac{\|T_n(x) - T_m(x)\|_\alpha^{\sim 1}}{\|x\|_\alpha^2} = 0,$$

which implies that for each $\alpha \in (0, 1]$ and $x \in X$,

$$\lim_{n,n\to\infty}\|T_n(x) - T_m(x)\|_\alpha^{\sim 1} = \lim_{n,n\to\infty}\|T_n(x) - T_m(x)\|_\alpha^{\sim 2} = 0.$$

From the fact that Y is a complete fuzzy normed space, for some y_x, $\lim_{n\to\infty} T_n(x) = y_x$ exists in Y.

Let for each $x \in X$, $\lim_{n\to\infty} T_n(x) = T(x)$. Trivially T is linear. Now for every $\alpha \in (0, 1]$,

$$\left| \|T_n(x)\|_\alpha^{*1} - \|T_m(x)\|_\alpha^{*1} \right| \le \|T_n(x) - T_m(x)\|_\alpha^{*1} \to 0 \text{ as } n, m \to \infty.$$

This implies that for all $\alpha \in (0, 1]$,

$$\left| \|T_n(x)\|_\alpha^{*1} - \|T_m(x)\|_\alpha^{*1} \right| \to 0 \text{ as } n, m \to \infty.$$

Thus for each $\alpha \in (0, 1]$, $\{\|T_n(x)\|_\alpha^{*1}\}$ is a Cauchy sequence of non-negative real number and so is convergent. Let for all $\alpha \in (0, 1]$, $\lim_{n\to\infty} \|T_n(x)\|_\alpha^{*1} = a_\alpha$.

Similarly $\{\|T_n(x)\|_\alpha^{*1}\}$ is a Cauchy sequence of non-negative real number for each $\alpha \in (0,1]$, so is convergent. Put $\lim_{n\to\infty} \|T_n(x)\|_\alpha^{*2} = b_\alpha$, $\alpha \in (0, 1]$. One can easily verify that $\{[a_\alpha, b_\alpha] : \alpha \in (0, 1]\}$ is a family of nested bounded closed interval of real numbers it generates a fuzzy real number, say η.

Now from

$$\|T(x)\|_\alpha^{\sim 1} = \lim_{n,m\to\infty} \|T_n(x)\|_\alpha^{\sim 1} \le \lim_{n\to\infty} \left(\|T_n\|_\alpha^{*1} \cdot \|x\|_\alpha^2 \right) = a_\alpha \cdot \|x\|_\alpha^2,$$

we have

$$\frac{\|T(x)\|_\alpha^{\sim 1}}{\|x\|_\alpha^2} \le a_\alpha, \forall x (\ne 0) \in X, \forall x \in (0,1]. \qquad (1)$$

Similarly

$$\frac{\|T(x)\|_\alpha^{\sim 2}}{\|x\|_\alpha^1} \le b_\alpha, \forall x (\ne 0) \in X, \forall x \in (0,1]. \qquad (2)$$

we know

$$\left\{\frac{\|T(x)\|_{\alpha}^{\sim 1}}{\|x\|_{\alpha}^{2}}, \frac{\|T(x)\|_{\alpha}^{\sim 2}}{\|x\|_{\alpha}^{1}}\right\} : \alpha \in (0,1)$$

generates the fuzzy interval $\|T(x)\|^{\sim} \ominus \| x \| \preceq \eta$, $\forall x (\neq 0) \in X$. This implies that $T \in B(X, Y)$.

We claim that $\|Tn - T\|^{*} \longrightarrow 0$ as $n \longrightarrow \infty$. For a given $\varepsilon > 0$, and each $\alpha \in (0, 1]$, there exists positive integer $N(\varepsilon, \alpha)$ such that for all n, m > $N(\varepsilon, \alpha)$, $\|T_n - T_m\|_{\alpha}^{*1} < \varepsilon$.

Thus for any $\alpha \in (0, 1]$ and $n, m \geq N(\varepsilon, \alpha)$,

$$\|T_n(x) - T_m(x)\|_{\alpha}^{*1} \leq \|T_n - T_m\|_{\alpha}^{*1} \cdot \|x\|_{\alpha}^{2} \leq \varepsilon \cdot \|x\|_{\alpha}^{2}.$$

So when $m \longrightarrow 1$, for every $\alpha \in (0, 1]$ and $n \geq N(\varepsilon, \alpha)$, we have

$$\|T_n(x) - T_m(x)\|_{\alpha}^{*1} \leq \varepsilon \cdot \|x\|_{\alpha}^{2},$$

which implies that

$$\bigvee_{0 \neq x \in X} \left\{\frac{\|T_n(x) - T_m(x)\|_{\alpha}^{*1}}{\|x\|_{\alpha}^{2}}\right\} \leq \varepsilon, \forall n \geq N(\varepsilon, \alpha).$$

Hence for $n \geq N(\varepsilon, \alpha)$ and $\alpha \in (0, 1]$, $\|T_n - T\|_{\alpha}^{*1} \leq \varepsilon$,

i.e. as $n \longrightarrow 1$ and $\alpha \in (0, 1]$ $\|T_n - T\|_{\alpha}^{*1} \to 0$.

Similarly we have $\|T_n - T\|_{\alpha}^{*2} \to 0$ as $n \longrightarrow \infty$ and $\alpha \in (0, 1]$. This follows that $\|T_n - T\|_{\alpha}^{*} \to 0$ as $n \longrightarrow \infty$, which implies that $(B(X, Y), \|.\|^{*})$ is a complete fuzzy normed space.

Example(2.3.2)[17] Let $(X, \|.\|)$ be an arbitrary Banach space, define

$$\|I\|_{\alpha}^{*1} = \sup_{0 \neq x \in X} \frac{\|x\|}{\|x\| / \alpha} = \alpha \quad \text{and} \quad \|I\|_{\alpha}^{*2} = \sup_{0 \neq x \in X} \frac{\|x\| / \alpha}{\|x\|} = \frac{1}{\alpha} .$$

Suppose δ is the fuzzy real number generated by $\{[\alpha, \frac{1}{\alpha}]:\alpha\in(0, 1]\}$.

So $\|SoT\|^* = \|S\|^* = \|T\|^* = \delta$.

On the other hand, $[\delta\ \Theta\delta]_\alpha = [\alpha^2, \frac{1}{\alpha^2}]$, and $[\delta\Theta\delta]_\alpha = [\alpha^2, \frac{1}{\alpha^2}], \alpha\in(0, 1]$.

But non of the relations $\alpha \leq \alpha^2$ and $\frac{1}{\alpha^2} \leq \frac{1}{\alpha}$ are valid. This shows that the relations $\delta \preceq \delta\ \Theta\ \delta$ and $\delta\ \Theta\ \delta \preceq \delta$ are not correct.

Theorem(2.3.3)[17] Let (X, ||.||) and (Y, $||.||^{\sim}$) be two fuzzy normed spaces. Then T∈B(X, Y) if and only if for every α∈ (0, 1], $T\in B((X, \|\cdot\|_\alpha^1), (Y, \|\cdot\|_\alpha^2))$ and $T \in B((X, \|\cdot\|_\alpha^{\sim 2}), (Y, \|\cdot\|_\alpha^2))$.

Proof Without loss of generality T is supposed to be non-null. Let for each α∈(0, 1], $T\in B((X, \|\cdot\|_\alpha^1), (Y, \|\cdot\|_\alpha^{\sim 2})) \cap B((X, \|\cdot\|_\alpha^2), (Y, \|\cdot\|_\alpha^{\sim 1}))$.

We show that T∈ B(X, Y). For α∈(0, 1], there exist $\delta_\alpha^1, \delta_\alpha^2 > 0$ such that for all $0 \neq x \in X$,

$$\frac{\|T(x)\|_\alpha^{\sim 2}}{\|x\|_\alpha^1} \leq \delta_\alpha^2, \text{ and } \frac{\|T(x)\|_\alpha^{\sim 1}}{\|x\|_\alpha^2} \leq \delta_\alpha^1,$$

since $T \in B((X, \|\cdot\|_\alpha^1), (Y, \|\cdot\|_\alpha^{\sim 2})) \cap B((X, \|\cdot\|_\alpha^2), (Y, \|\cdot\|_\alpha^{\sim 1}))$.

For each $\alpha \in (0, 1]$, define

$$\delta_\alpha^{*2} = \inf\left\{\delta_\alpha^2\ \frac{\|T(x)\|_\alpha^{\sim 2}}{\|x\|_\alpha^1} \leq \delta_\alpha^2,\ \forall x(\neq 0) \leq X\right\},$$

and

$$\delta_\alpha^{*1} = \inf\left\{\delta_\alpha^1\ \frac{\|T(x)\|_\alpha^{\sim 1}}{\|x\|_\alpha^1} \leq \delta_\alpha^1,\ \forall x(\neq 0) \leq X\right\},$$

So for all $0\neq x \in X$, by definition of infimum,

$\frac{\|T(x)\|_\alpha^{\sim 2}}{\|x\|_\alpha^1} \le \delta_\alpha^2$, and $\frac{\|T(x)\|_\alpha^{\sim 1}}{\|x\|_\alpha^2} \le \delta_\alpha^1$. We know $\{[\delta_\alpha^{*1}, \delta_\alpha^{*2}] : \alpha \in 2\ (0, 1]\}$ is a family of nested bounded closed intervals of real numbers, so using Proposition (2.2.4), this family generates a positive fuzzy interval, $\eta \succ 0$, and by Proposition (2.2.5), $\|T(x)\|^{\sim} \Theta \|x\| \preceq \eta$.

This means that $T \in B(X, Y)$. The converse is trivial.

This theorem has the following useful corollary which shows that every linear operator from a finite dimensional fuzzy linear space is fuzzy continuous. For its proof it is enough to use the previous theorem and the ordinary version of this corollary in functional analysis.

Corollary(2.3.4)[17] Let $T : X \longrightarrow Y$ be a linear operator where $(X, \|.\|)$ and $(Y, \|.\|^{\sim})$ are fuzzy normed linear spaces. If X has finite dimension, then T is fuzzy bounded (so is fuzzy continuous).

We present some counterexamples. The first is a counterexample for the uniform boundedness principle in this fuzzy structure. Before stating this example we need the following definition.

Definition(2.3.5)[17] Let $(X, \| . \|)$ and $(Y, \| . \|^{\sim})$ be two fuzzy normed linear spaces. A family $\{T_n\} \subseteq B(X, Y)$ is called point-wise bounded if for every $x(\neq 0) \in X$, there exists fuzzy number $\delta_x \in F^+$, $\delta_x \succ 0$, such that for all $n > 0$, $\|T(x)\|^{\sim} \Theta \|x\| \preceq \delta_x$, and is said to be uniformly bounded if there exists fuzzy number $\delta \in F^+$, $\delta \succ 0$, such that for each $n > 0$ and $x(\neq 0) \in X$, $\|T(x)\|^{\sim} \Theta \|x\| \preceq \delta$.

For a sequence $\{T_n\}$ of fuzzy bounded operators from a fuzzy Banach space X to a fuzzy Banach space Y, it is expected that the point-wise and uniformly boundedness of this sequence are equivalent (a fuzzy version of uniform boundedness principle), but the following example shows that this is not true in general.

Example(2.3.6)[17] Consider X = l^1, the set of all real valued sequences whose series is absolutely convergent, we know X with $||.||_\infty$ is Banach but with $||.||_1$ is not Banach and $||.||_\infty \leq ||.||_1$. Now suppose $||x||$ is the fuzzy norm generated by the nested family $\{|\alpha||x||_1, \frac{1}{\alpha}||x||_1] : \alpha \in (0, 1]\}$ of intervals. Let Y = R with the fuzzy norm generated by the family $\{[|x|, |x|] : \alpha \in (0, 1]\}$. One can see that (X, $||.||$) and (Y, $|.|$) are two fuzzy Banach spaces.

Define $T_n : X \longrightarrow Y$ by $T_n\left((x_k)_k\right) = \sum_{k=1}^{n} x_k$.

From Theorem (2.3.2), $T_n \in B(X, Y)$, $n \in N$. We are going to show that $\{T_n\}$ is point-wise bounded but is not uniformly bounded. Let $0 \neq x = (x_k)_{k\in N} \in X$. If δ_x is the fuzzy number generated by the family $\{[\alpha, \frac{\|x\|_1}{\alpha\|x\|_\infty}] : \alpha \in (0, 1]\}$, then for any n > 0, $\|T(x)\|\tilde{\ }\Theta\|x\| \preceq \delta_x$.

Now in contrary suppose $\{T_n\}$ is uniformly bounded, so there exists $\delta \in F^+$ such that for each $0 \neq x \in X$, and n > 0, $\|T(x)\|\tilde{\ }\Theta\|x\| \preceq \delta$.

Thus for every $\alpha \in (0, 1]$ and $x = (x_k)_{k\in N}$, with $x_1 = ... = x_n = 1$ and $x_k = 0$, k > n, $\frac{\|T(x)\|_\alpha^2}{\|x\|_\alpha^1} \leq \delta_\alpha^2$, this implies that for each $n \in N$, $\frac{n}{1} \leq \delta_\alpha^2$.

Hence $\delta_\alpha^2 = \infty$, which contradicts the definition of fuzzy real number. From inverse mapping theorem in ordinary case, we know if X, Y are ordinary Banach spaces and $T \in B(X, Y)$ is a bijection map, then its inverse belongs to B(Y,X). The following example shows that this is not valid in this fuzzy setting.

Example(2.3.7)[17] Suppose $||.||^1$ and $||.||_2$ are two ordinary norm on a vector space X, such that (X, $||.||_1$) is not Banach space and (X, $||.||_2$) is Banach space, for which

$$||x||_1 \le ||x||_2, \ \forall x \in X,$$

(For example let X = C[0, 1], with $||.||_1$ and $||.||_\infty$). For each x, trivially $\{[||x||_1, \frac{\|x\|_2}{\alpha}]: \alpha \in (0, 1]\}$ is a family of nested closed intervals which generates a fuzzy interval |||x|||. Also consider the fuzzy norm $|||x|||^{\sim}$ on X generated by the family $\{[||x||_2, ||x||_2] : \alpha \in (0, 1]\}$. One can see that (X, |||.|||) and (X, |||.|||~) are fuzzy Banach spaces. The identity mapping I : (X, $|||.|||^{\sim}$) ⟶ (X, |||.|||) is a fuzzy bounded linear operator, since if δ is the fuzzy number generated by the family $\{[1, \frac{1}{\alpha}] : \alpha \in (0, 1]\}$, then

$$|||x||| \ \Theta \ |||x|||^{\sim} \preceq \delta.$$

Now we show that I^{-1} is not bounded. Otherwise, so there exists $\eta \in F^+$ such that for each $x \in X$,

$$r(x) := |||x|||^{\sim} \ \Theta \ |||x||| \preceq \eta.$$

It means that for each $\alpha \in (0, 1]$, if $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$, and $[r(x)]_\alpha = [r(x)_\alpha^1, r(x)_\alpha^2]$, then $r(x)_\alpha^1 \le \eta_\alpha^1$ and $r(x)_\alpha^2 \le \eta_\alpha^2$. Now let $\alpha, \beta \in (0, 1]$ with $\alpha > \beta$. From the assertion before Proposition (2.2.4), for each $x \in X$,

$$\left[\frac{\|x\|_2}{\frac{1}{\beta}\|x\|_2}, \frac{\|x\|_2}{\|x\|_1}\right] \subseteq \left[r(x)_\alpha^1, r(x)_\alpha^2\right].$$

Hence

$$\frac{\|x\|_2}{\|x\|_1} \le r(x)_\alpha^2 \le \eta_\alpha^2.$$

This shows that $||x||_2 \leq \eta_\alpha^2 ||x||_1$, on the other hand by our hypothesis $||x||_1 \leq ||x||_2$. This means that $||.||_1$ and $||.||_2$ are equivalent which is a contradiction.

Theorem(2.3.8)[17] Every two fuzzy norms on a finite dimensional vector space X are semi-equivalent.

Proof Let $||.||_1$ and $||.||_2$ be two fuzzy norms on a finite dimensional vector space X and $\alpha \in (0, 1]$. Define $I : (X, ||.||_{1\alpha}^2) \longrightarrow (X, ||.||_{2\alpha}^1)$ and $I : (X, ||.||_{1\alpha}^1) \longrightarrow (X, ||.||_{2\alpha}^2)$ by $I(x) = x$. Thus for all $\alpha \in (0, 1]$ there exist positive real numbers δ_α^1 and δ_α^2 such that

$$\frac{\|x\|_{2\alpha}^1}{\|x\|_{1\alpha}^2} \leq \delta_\alpha^1, \forall x(\neq 0) \in X,$$

and

$$\frac{\|x\|_{2\alpha}^2}{\|x\|_{1\alpha}^1} \leq \delta_\alpha^2, \forall x(\neq 0) \in X,$$

For each $\alpha \in (0, 1]$, define

$$\delta_\alpha^{*1} = \inf\left\{\delta_\alpha^1 : \frac{\|x\|_{2\alpha}^1}{\|x\|_{1\alpha}^2} \leq \delta_\alpha^1, \forall x(\neq 0) \in X\right\},$$

and

$$\delta_\alpha^{*2} = \inf\left\{\delta_\alpha^2 : \frac{\|x\|_{2\alpha}^2}{\|x\|_{1\alpha}^1} \leq \delta_\alpha^2, \forall x(\neq 0) \in X\right\},$$

Now from the fact that $\{[\delta_\alpha^{*1}, \delta_\alpha^{*2}] : \alpha \in (0, 1]\}$ is a family of nested bounded closed interval of real numbers, using Proposition (2.2.4), this family generates a positive fuzzy interval δ. By Proposition (2.2.5), $\delta \succ 0$ and $||x||_2 \ominus ||x||_1 \succeq \delta, \forall x(\neq 0) \in X$.

Similarly there exists positive fuzzy interval η such that for all $x(\neq 0) \in X$, $\|x\|_1 \ominus \|x\|_2 \preceq \eta$, which implies that $\|.\|_1$ and $\|.\|_2$ are semi-equivalent.

Hahn-Banach Theorem(2.3.9)[17]

Bag and Samanta established a Hahn-Bancah theorem for the strong dual of fuzzy normed linear space. Here we prove it for the weak dual of fuzzy linear spaces. Also some interesting consequences of this theorem are established.We recall that a fuzzy bounded linear operator from a fuzzy normed space X into R, with the fuzzy norm $\|.\|^{\sim}$ generated by the family $\{[|x|, |x|] : \alpha \in (0, 1]\}$ is called a fuzzy bounded functional, and the set of all such a functional is denoted by X^*.

Theorem(2.3.10)[17] Let X be a fuzzy normed linear space and Z be a subspace of X. If f is a fuzzy bounded linear functional on $(Z,\|.\|)$, then for each $\alpha \in (0,1]$, there exists a pair of linear functionals f^1_α and f^2_α over $(X,\|.\|^2_\alpha)$ and $(X, \|.\|^1_\alpha)$, respectively, such that for all $x \in Z$, $f^1_\alpha(x) = f^2_\alpha(x) = f(x)$ and for each

$\alpha \in (0, 1]$, $\|f^1_\alpha\|^{*1}_\alpha = \|f\|^{*1}_\alpha$, $\|f^2_\alpha\|^{*2}_\alpha = \|f\|^{*2}_\alpha$.

Proof By definition, there exists $\delta \in F^+$, $\delta \succ \bar{0}$, such that for all $(0 \neq)x \in Z$, $\|f(x)\|^{\sim} \ominus \|x\| \preceq \eta$.

So for each $\alpha \in (0, 1]$,

$$\frac{|f(x)|}{\|x\|^2_\alpha} \leq \delta^1_\alpha, \frac{|f(x)|}{\|x\|^1_\alpha} \leq \delta^2_\alpha.$$

Thus

$$\|f\|^{*1}_\alpha = \sup\left\{\frac{|f(x)|}{\|x\|^1_\alpha} : (0 \neq)x \in Z\right\} \leq \delta^1_\alpha.$$

and

$$\|f\|_{\alpha}^{*2} = \sup\left\{\frac{|f(x)|}{\|x\|_{\alpha}^{2}} : (0 \neq) x \in Z\right\} \leq \delta_{\alpha}^{2}.$$

This means that f is in the dual of (Z, $||.||_{\alpha}^{1}$) and (Z, $||.||_{\alpha}^{2}$), with the norms $||.||_{\alpha}^{*2}$ and $||.||_{\alpha}^{*1}$, respectively. Now using ordinary Hahn-Banach theorem, there exist f_{α}^{1} and f_{α}^{2} in the dual of (X, $||.||_{\alpha}^{2}$) and (X, $||.||_{\alpha}^{1}$, respectively, such that

$$||f_{\alpha}^{1}||_{\alpha}^{*1} = ||f||_{\alpha}^{*1} \text{ and } ||f_{\alpha}^{2}||_{\alpha}^{*2} = ||f||_{\alpha}^{*2},$$

and for all $x \in Z$, $f_{\alpha}^{1}(x) = f_{\alpha}^{2}(x) = f(x)$.

Chapter 3

Some Topological and Algebraic Properties of α-Level Subsets' Topology of a Fuzzy Subset

Section (3-1) Fuzzy Anti-Norm and Fuzzy α-Anti-Convergence

During the last few years there is a growing interest in the extension of fuzzy set theory which is a useful tool to describe the situation in which data are imprecise or vague or uncertain. Fuzzy set theory handle the situation by attributing a degree of membership to which a certain object belongs to a set. It has a wide range of application in the field of population dynamics, chaos control, computer programming, medicine etc.

The concept of fuzzy set theory was first introduced by Zadeh in 1965 and thereafter, the concept of fuzzy set theory applied on different branches of pure and applied mathematics in different ways, by several authors. The concept of fuzzy norm was introduced by Katsaras in 1984. In 1992, Felbin introduced the idea of fuzzy norm on a linear space. Cheng–Moderson introduced another idea of fuzzy norm on a linear space whose associated metric is same as the associated metric of Kramosil–Michalek. In 2003, Bag and Samanta modified the definition of fuzzy norm of Cheng–Moderson and established the concept of continuity and boundednes of a linear operator with respect to their fuzzy norm.

Later on, Jebril and Samanta introduced the concept of fuzzy antinorm on a linear space depending on the idea of fuzzy anti norm, introduced by Bag and Samanta. The motivation of introducing fuzzy anti-norm is to study fuzzy set theory with respect to the non-membership function. It is useful in the process of decision making.

We generalize the definition of fuzzy anti-norm on a linear space. Later on we prove Riesz lemma and some important properties of finite dimensional fuzzy anti-normed linear space. Also, we define fuzzy α- anti-convergence, fuzzy α-anti-Cauchy sequence, fuzzy α-anti-completeness and study the relations among them.

Preliminaries(3.1.1)[6]

This section contains some basic definition and preliminary results which will be needed.

Definition(3.1.2)[6] A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-conorm if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a, \forall a \in [0, 1]$,

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0, 1]$.

A few examples of continuous t-conorm are $a \diamond b = a + b - ab$, $a \diamond b = \max\{a, b\}$, $a \diamond b = \min\{a + b, 1\}$.

Definition(3.1.3)[6]Let X be a linear space over F (field of real/complex numbers). Let N^* be a fuzzy subset of $X \times R$ such that for all $x, y \in X$ and $c \in F$

(N^*1) $\forall t \in R$ with $t \leq 0$, $N^*(x, t) = 1$,

(N^*2) $\forall t \in R$ with $t > 0$, $N^*(x, t) = 0$ if and only if $x = \theta$,

(N^*3) $\forall t \in R$ with $t > 0$, $N^*(cx, t) = N^*(x, \frac{t}{|c|})$ if $c \neq 0$,

(N^*4) $\forall s, t \in R$ with $N^*(x + y, s + t) \leq \max\{N^*(x, s), N^*(y, t)\}$,

(N^*5) $N^*(x, \cdot)$ is a non-increasing of $t \in R$ and $\lim_{t\to\infty} N^*(x, t) = 0$.

Then N^* is called a B-S-fuzzy antinorm on X.

We assume that

(N^*6) For all $t \in R$ with $t > 0$, $N^*(x, t) < 1$ implies $x = \theta$.

Definition(3.1.4)[6] Let (U,N^*) be a B-S-fuzzy antinormed linear space. A sequence $\{x_n\}^{n\in N}$ in U is said to converge to $x \in U$ if given $t > 0$, $r \in (0, 1)$ there exists an integer $n_0 \in N$ such that

$$N*(x_n - x, t) < r, \forall n \geq n_0.$$

Definition(3.1.5)[6] Let (U,N^*) be a B-S-fuzzy antinormed linear space. A sequence $\{x_n\}_{n\in N}$ in U is said to be Cauchy sequence if for given $t > 0$, $r \in (0, 1)$ there exists an integer $n_0 \in N$ such that

$$N^*(x_{n+p} - x_n, t) < r \ \forall n \geq n_0, p = 1, 2, 3, \dots .$$

Definition(3.1.6)[6] A subset A of a B-S-fuzzy antinormed linear space (U,N^*) is said to be bounded if and only if there exist $t>0$, $r \in (0, 1)$ such that

$$N^*(x, t) < r, \forall x \in A.$$

Definition(3.1.7)[] A subset A of a B-S-fuzzy antinormed linear space (U,N^*) is said to be compact if any sequence $\{x_n\}_{n\in N}$ in A has a subsequence converging to an element of A.

Definition(3.1.8)[6] Let (U,N^*) be a B-S-fuzzy antinormed linear space. A subset B of U is said to be closed if any sequence $\{x_n\}_{n\in N}$ in B converges to $x \in B$, that is

$$\lim_{n\to\infty} N^*(x_n - x, t) = 0, \forall t > 0 \Rightarrow x \in B.$$

Fuzzy anti-normed linear space(3.1.9)[6]

The definition of B-S-fuzzy antinorm is modified, and after modification it will be termed as fuzzy antinorm with respect to a t-conorm $\diamond$. Thereafter some important results will be deduced.

Definition(3.1.10)[6] Let V be linear space over the field F(=RorC). A fuzzy subset ν of $V \times R$ is called a fuzzy antinorm on V with respect to a t-conorm $\diamond$ if and only if for all $x, y \in V$

(i) $\forall t \in R$ with $t \leq 0$, $\nu(x, t) = 1$;

(ii) $\forall t \in R$ with $t > 0$, $\nu(x, t) = 0$ if and only if $x = \theta$;

(iii) $\forall t \in R$ with $t > 0$, $\nu(cx, t) = \nu(x, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$;

(iv) $\forall s, t \in R$ with $\nu(x + y, s + t) \leq \nu(x, s) \diamond \nu(y, t)$;

(v) $\lim_{t\to\infty} \nu(x, t) = 0$.

The Definition (3.1.11) is more general than the Definition (3.1.9); since, in (N^*4) instead of maximum function we have used more generalized function, conorm function and in the condition (N^*5) it is used that $N^*(x, \cdot)$ is nonincreasing function of $t(\in R)$, which is redundant and later on it will be deduced.

Remark(3.1.11)[6] Let ν be a fuzzy anti-norm on V with respect to a t-conorm $\diamond$, then $\nu(x, t)$ is non-increasing with respect to t for each $x \in V$.

Proof. Let $t < s$. Then $k = s - t > 0$ and we have

$$\nu(x, t) = \nu(x, t) \diamond 0 = \nu(x, t) \diamond \nu(0, k) \geq \nu(x, s).$$

Hence the proof.

Definition (3.1.12)[6] If $A^* = \{((x, t), \nu(x, t)) : (x, t) \in V \times R\}$ is a fuzzy antinorm on a linear space V with respect to a t-conorm $\diamond$ over a field F, then (V,A^*) is called a fuzzy antinormed linear space with respect to the t-conorm $\diamond$ over the field F.

We further assume that for any fuzzy anti-normed linear space (V,A^*) with respect to a t-conorm $\diamond$,

(vi) $\nu(x, t) < 1, \forall t > 0 \Rightarrow x = \theta$.

(vii) $\nu(x, \cdot)$ is a continuous function of R and strictly decreasing on the subset $\{t : 0 < \nu(x, t) < 1\}$ of R.

(viii) $a \diamond a = a, \forall a \in [0, 1]$.

Example (3.1.13)[6] Let (V, || · ||) be a normed linear space and consider $a \diamond b = a + b - ab$. Define $\nu : V \times R \to [0, 1]$ by

$$\nu(x,t)=\begin{cases}0, & \text{if } t>\|x\|,\\ 1, & \text{if } t\le\|x\|.\end{cases}$$

Then ν is a fuzzy antinorm on V with respect to the t-conorm ⋄ and (V, ν) is a fuzzy anti-normed linear space with respect to the t-conorm ⋄.

(i) $\forall x \in V$ and $\forall t \in R$, $t \le 0$ we have $\nu(x, t) = 1$.

(ii) $\forall t \in R$, $t > 0$ we have $\nu(\theta, t) = 0$. Again

$\nu(x, t) = 0, \forall t > 0 \Leftrightarrow ||x|| < t, \forall t(> 0) \in R \Leftrightarrow ||x|| = 0 \Leftrightarrow x = \theta$.

(iii) $\nu(cx, t)=0 \Leftrightarrow t>||cx|| \Leftrightarrow t > |c|||x|| \Leftrightarrow \frac{t}{|c|} \le ||x|| \Leftrightarrow \nu\left(x, \frac{t}{|c|}\right)= 0$.

$\nu(cx, t) = 1 \Leftrightarrow t \le ||cx|| \Leftrightarrow t \le |c|||x|| \Leftrightarrow \frac{t}{|c|} \le ||x|| \Leftrightarrow \nu\left(x, \frac{t}{|c|}\right)= 1$.

(iv) $\nu(x, s) \diamond \nu(y, t) = \nu(x, s) + \nu(y, t) - \nu(x, s)\nu(y, t)$.

If $s > ||x||$ and $t > ||y||$ then $\nu(x+ y, s + t) = 0$, since $s + t > ||x|| + ||y||$ and $\nu(x, s) \diamond \nu(y, t) = 0 + 0 - 0 = 0$. So, $\nu(x + y, s + t) = \nu(x, s) \diamond \nu(y, t)$.

If $s > ||x||$ and $t \le ||y||$ then $\nu(x, s) \diamond \nu(y, t) = 0 + 1 - 0 = 1$.

If $s \le ||x||$ and $t > ||y||$ then $\nu(x, s) \diamond \nu(y, t) = 1 + 0 - 0 = 1$.

If $s \le ||x||$ and $t \le ||y||$ then $\nu(x, s) \diamond \nu(y, t) = 1 + 1 - 1 = 1$.

Therefore in any of the above three cases

$$\nu(x, s) \diamond \nu(y, t) = 1 \ge \nu(x + y, s + t).$$

Thus

$$\nu(x + y, s + t) \le \nu(x, s) \diamond \nu(y, t).$$

(v) From the definition it is clear that $\lim_{t\to\infty} \nu(x, t) = 0$. Thus ν is a fuzzy antinorm on V with respect to the t-conorm ⋄ and (V, ν) is a fuzzy antinormed linear space with respect to the t-conorm ⋄.

Note(3.1.14)[6] The above example satisfes the condition (vi) but does not satisfy the condition (vii).

Example(3.1.15)[6] Let (V, || · ||) be a normed linear space and consider a ⋄ b = min{a + b, 1}. Define ν : V × R → [0, 1] by

$$\nu(x,t)=\begin{cases}0, & \text{if } t>\|x\|, t>0,\\ \frac{\|x\|}{t+\|x\|}, & \text{if } t\geq\|x\|, t>0\\ 1, & \text{if } t\leq 0.\end{cases}$$

Then ν is a fuzzy antinorm on V with respect to the t-conorm ⋄ and (V, ν) is a fuzzy anti-normed linear space with respect to the t-conorm ⋄.

(i) From the definition we have ν(x, t) = 1 if $t \leq 0$, ∀t ∈ R.

(ii) If t > 0 and t > ||x|| then

$$\nu(x, t) = 0 \Leftrightarrow \|x\| < t,\ \forall t(>0) \in R \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta.$$

If t > 0 and $t \leq \|x\|$ then

$$\nu(x, t) = 0 \Leftrightarrow \frac{\|x\|}{t+\|x\|} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta.$$

(iii) $\nu(cx, t)=0 \Leftrightarrow t > \|cx\| \Leftrightarrow t > |c|\|x\| \Leftrightarrow \frac{t}{|c|} > \|x\| \Leftrightarrow \nu\left(x, \frac{t}{|c|}\right)=0.$

$$\nu(cx, t) = \frac{\|cx\|}{t+\|cx\|} \Leftrightarrow t \leq \|cx\| \Leftrightarrow \frac{t}{|c|} \leq \|x\|$$

$$\Leftrightarrow \nu\left(x, \frac{t}{|c|}\right) = \frac{\|x\|}{\frac{t}{|c|}+\|x\|} = \frac{\|cx\|}{t+\|cx\|}.$$

(iv) ν(x, s) ⋄ ν(y, t) = min{ν(x, s) + ν(y, t),1}. If $\|x\| \geq s$ and $\|y\| \geq t$ then

$$\nu(x, s) + \nu(y, t) = \frac{\|x\|}{s+\|x\|} + \frac{\|y\|}{t+\|y\|}$$

$$= \frac{\left(t\|x\| + \|x\|\ \|y\| + s\|y\|\right) + \|x\|\ \|y\|}{\left(t\|x\| + \|x\|\ \|y\| + s\|y\|\right) + st} \geq 1 \text{ since } \|x\|\ \|y\| \geq st.$$

In this case $\nu(x, s) \diamond \nu(y, t) = 1 \geq \nu(x + y, s + t)$.

If $\|x\| \geq s$ and $\|y\| < t$ then either $\|x + y\| \geq s + t$ or $\|x + y\| < s + t$. Now,

$$\nu(x, s) + \nu(y, t) = \frac{\|x\|}{s + \|x\|} + 0 < 1.$$

Hence

$$\nu(x, s) \diamond \nu(y, t) = \frac{\|x\|}{s + \|x\|}.$$

If $\|x + y\| \geq s + t$ then

$$\nu(x + y, s + t) - \nu(x, s) \diamond \nu(y, t) = \frac{\|x + y\|}{s + t + \|x + y\|} - \frac{\|x\|}{s + \|x\|}$$

$$\leq \frac{\|x\| + \|y\|}{s + t + \|x + y\|} - \frac{\|x\|}{s + \|x\|} = \frac{s\|y\| - t\|x\|}{\left(s + t + \|x\| + \|y\|\right)\left(s + \|x\|\right)}$$

$$< \frac{st - t\|x\|}{\left(s + t + \|x\| + \|y\|\right)\left(s + \|x\|\right)}, \text{ since } \|y\| < t,$$

$$\leq 0, \text{ since } s \leq \|x\| \Rightarrow st < t\|x\|.$$

Therefore, $\nu(x + y, s + t) < \nu(x, s) \diamond \nu(y, t)$.

If kx + yk < s + t then

$$\nu(x + y, s + t) = 0 \leq \frac{\|x\|}{s + \|x\|} = \nu(x, s) \diamond \nu(y, t).$$

If $\|x\| < s$ and $\|y\| \geq t$ then in the similar manner (as in the case when $\|x\| \geq s$ and $\|y\| < t$) we can show that $\nu(x + y, s + t) \leq \nu(x, s) \diamond \nu(y, t)$.

If $\|x\| < s$ and $\|y\| < t$ then $\nu(x, s) + \nu(y, t) = 0 + 0 < 1$. Therefore,

$\nu(x, s)\diamond\nu(y, t) = 0$. Also $||x+y|| \le ||x||+||y|| < s + t$ and hence $\nu(x+y, s+t)= 0$. Hence $\nu(x + y, s + t) = \nu(x, s) \diamond \nu(y, t)$. Thus, in any case $\nu(x + y, s + t) \le \nu(x, s) \diamond \nu(y, t)$.

(v) If $t > ||x||$ then from the definition it is clear that $\lim_{t\to\infty} \nu(x, t) = 0$.

If $x \ne \theta$ and $t \le ||x||$ then

$$\lim_{t\to\infty} \nu(x, t) = \lim_{t\to\infty} \nu(\theta, t) = \lim_{t\to\infty} \tfrac{0}{t}.$$

Hence

$$\lim_{t\to\infty} \nu(x, t) = 0 \ \forall x \in V.$$

Thus ν is a fuzzy antinorm on V with respect to the t-conorm $\diamond$ and (V, ν) is a fuzzy anti-normed linear space with respect to the t-conorm $\diamond$.

Note (3.1.16)[6] The above example does not satisfy the conditions (vi) and (vii).

Example(3.1.17)[6] Let $(V, ||\cdot||)$ be a normed linear space and consider $a \diamond b = \max\{a, b\}$. Define $\nu : V \times R \to [0, 1]$ by

$$\nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|}, & \text{if } t > 0, \\ 1, & \text{if } t \le 0. \end{cases}$$

Note(3.1.18)[6] The above example does not satisfy the condition (vi) and satisfies the condition (vii).

Example (3.1.19)[6] Let $(V, ||\cdot||)$ be a normed linear space and consider $a \diamond b = \min\{a + b, 1\}$. Define $\nu : V \times R \to [0, 1]$ by

$$\nu(x,t) = \begin{cases} \frac{\|x\|}{2t-\|x\|}, & \text{if } t > \|x\|, \\ 1, & \text{if } t \le \|x\|. \end{cases}$$

Then ν satisfies all conditions of Definition (3.1.11). Therefore, ν is a fuzzy antinorm on V with respect to the t-conorm $\diamond$ and (V, ν) is a fuzzy antinormed linear space with respect to the t-conorm $\diamond$.

Theorem(3.1.20)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄ satisfying (vi) and (viii). Then for any $\alpha \in (0, 1)$ the function $\|x\|_\alpha^*: X \to [0,\infty)$ defined as

(ix) $\|x\|_\alpha^* = \wedge\{t > 0 : \nu(x, t) \leq 1 - \alpha\}$, $\alpha \in (0, 1)$ is a norm on V.

Proof.

(i) For $x \in V$, $\nu(x, t) = 1$ for $t \leq 0 \Rightarrow V \{t > 0 : \nu(x, t) \leq 1-\alpha\} \geq 0$, $\alpha \in (0, 1) \Rightarrow \|x\|_\alpha^* \geq 0$, $\alpha \in (0, 1)$.

(ii) $\|x\|_\alpha^* = 0 \Rightarrow \nu(x, t) \leq 1 - \alpha < 1$, $\forall t \in R$, $t > 0 \Rightarrow x = \theta$, [by (vi)].

Conversely, $x =\theta \Rightarrow \nu(x, t) = 0$, $\forall t > 0 \Rightarrow V \{t >0: \nu(x, t) \leq 1-\alpha\}= 0$, $\forall \alpha \in (0, 1) \Rightarrow \|x\|_\alpha^* = 0$.

(iii) If $c \neq 0$ then

$$\|cx\|_\alpha^* = \wedge\{s > 0 : \nu(cx, s) \leq 1 - \alpha\}$$
$$= \wedge\left\{s > 0 : \nu\left(x, \tfrac{s}{|c|}\right) \leq 1 - \alpha\right\}$$
$$= \wedge \{|c|t > 0 : \nu(x, t) \leq 1 - \alpha\}$$
$$=\wedge|c|\{t > 0 : \nu(x, t) \leq 1 - \alpha\} = |c|\|x\|_\alpha^*.$$

If $c = 0$ then $\|cx\|_\alpha^* = \|\theta\|_\alpha^* = 0 = 0$. $\|x\|_\alpha^* = |c|\ \|x\|_\alpha^*$.

(iv) $\|x\|_\alpha^* + \|y\|_\alpha^* =$

$$\wedge\{t > 0 : \nu(x, t) \leq 1 - \alpha\} + \wedge\{s > 0 : \nu(y, s) \leq 1 - \alpha\},\ \forall \alpha \in (0, 1)$$
$$\geq \wedge\{t + s > 0 : \nu(x, t) \leq 1 - \alpha,\ \nu(y, s) \leq 1 - \alpha\}$$
$$\geq \wedge\{t + s > 0 : \nu(x + y, t + s) \leq 1 - \alpha \text{ [by (viii)]} = \|x + y\|_\alpha^*.$$

Hence, $\{\|\cdot\|_\alpha^*\}$ is a norm on V.

Remark(3.1.21)[6] The norm defined above is more general than the norm defined; since instead of $\nu(x, t) <\alpha$ we write $\nu(x, t) \leq 1- \alpha$.

Theorem(3.1.22)[6] Let (V,A*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄. If $\alpha_1 \le \alpha_2$, then $\|x\|^*_{\alpha_1} \le \|x\|^*_{\alpha_2}$ i.e., $\{\|\cdot\|^*_{\alpha} : \alpha \in (0, 1)\}$ is an increasing family of norms on V .

Proof $\alpha_1 \le \alpha_2$ we have

$$\{t > 0 : \nu(x, t) \le 1 - \alpha_2\} \subset \{t > 0 : \nu(x, t) \le 1 - \alpha_1\}$$

$$\Rightarrow \wedge\{t > 0 : \nu(x, t) \le 1 - \alpha 2\} \ge \wedge\{t > 0 : \nu(x, t) \le 1 - \alpha 1\}$$

$$\Rightarrow \|x\|^*_{\alpha_2} \ge \|x\|^*_{\alpha_1}.$$

In the following theorem we describe another one equivalent expression for ν, which will be useful to describe Riesz theorem in fuzzy environment.

Theorem (3.1.23)[6] Let (V,A*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄ satisfying (vi), (vii), (viii) and let $\nu' : V \times R \to [0, 1]$ be defined by

$$\text{(x)}\quad \nu'(x,t) = \begin{cases} \wedge\{1-\alpha : \|x\|^*_{\alpha} \le t\}, & \text{if } (x,t) \ne (\theta,0), \\ 1, & \text{if } (x,t) = (\theta,0). \end{cases}$$

Then $\nu' = \nu$, where $\|x\|^*_{\alpha}$ is a increasing family of norms given by (ix).

To prove this theorem we use the following lemma.

Lemma(3.1.24)[6] Let (V,A*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄ satisfying (vi), (vii), (viii) and $\{\|\cdot\|^*_{\alpha} \in (0, 1)\}$ be increasing family of norms of V, defined by (ix). Then for $x_0(\ne \theta) \in V$, $\alpha \in (0, 1)$ and $s(> 0) \in R$,

$$\|x_0\|^*_{\alpha} = s \Leftrightarrow \nu(x_0, s) = 1 - \alpha.$$

Proof. Let $\|x_0\|_\alpha^* = s$, then $s > 0$. Then there exists a sequence $\{s_n\}_{n\in N}$, $s_n > 0$ such that $\nu(x_0, s_n) \le 1 - \alpha$, for all $n \in N$ and $s_n \to s$ as $n \to \infty$.

Therefore

$$\lim_{n\to\infty} \nu(x_0, s_n) \le 1 - \alpha \Rightarrow \nu(x_0, \lim_{n\to\infty} s_n) \le 1 - \alpha \text{ by (vii)}$$

$$\Rightarrow \nu(x_0, \|x_0\|_\alpha^*) \le 1 - \alpha, \forall\alpha \in (0, 1).$$

Let $\alpha \in (0, 1)$, $x_0(\neq \theta) \in V$ and $s = \|x_0\|_\alpha^* = \wedge\{t : \nu(x_0, t) \le 1 - \alpha\}$. Since $\nu(x, \cdot)$ is continuous (by (vii)) we have

$$\nu(x_0, s) \le 1 - \alpha. \quad (1)$$

If possible, let $\nu(x_0, s) < 1 - \alpha$, then by (vii) there exists $s' > s$ such that $\nu(x_0, s') < \nu(x_0, s) < 1 - \alpha$, which is impossible since $s = \wedge\{t : \nu(x_0, t) \le 1 - \alpha\}$. Thus

$$\nu(x_0, s) \ge 1 - \alpha. \quad (2)$$

From (1) and (2) it follows that $\nu(x_0, s) = 1 - \alpha$. Thus

$$\|x_0\|_\alpha^* = s \Rightarrow \nu(x_0, s) = 1 - \alpha. \quad (3)$$

Next, if $\nu(x_0, s) = 1 - \alpha$, $\alpha \in (0, 1)$ then by (vii)

$$\|x_0\|_\alpha^* = \wedge \{t : \nu(x_0, t) \le 1 - \alpha\} = s. \quad (4)$$

Hence, from (3) and (4), we have for $\alpha \in (0, 1)$, $x(\neq \theta) \in V$ and for $s > 0$, $\|x_0\|_\alpha^* = s \Leftrightarrow \nu(x_0, s) = 1 - \alpha$.

Proof of the main theorem. Let $(x_0, t_0) \in V \times R$. To prove this theorem, we consider the following cases:

Case 1: For any $x_0 \in V$ and $t \le 0$, $\nu(x_0, t_0) = \nu'(x_0, t_0) = 1$.

Case 2: If $x_0 = \theta$, $t_0 > 0$. Then $\nu(x_0, t_0) = \nu'(x_0, t_0) = 0$.

Case 3: $x_0 \neq \theta$, $t_0(> 0) \in R$ such that $\nu(x_0, t_0) = 1$.

we have, $\nu(x_0, \|x\|_\alpha^*) = 1-\alpha$ for all $\alpha \in (0, 1)$. Since $\nu(x_0, t_0) = 1 > 1-\alpha$ it follows that $\nu(x_0, \|x\|_\alpha^*) \leq 1-\alpha < \nu(x_0, t_0)$ and since $\nu(x_0, \cdot)$ is strictly non increasing $t_0 < \|x_0\|_\alpha^*$, $\forall \alpha \in (0, 1)$. So, $\nu'(x_0, t_0) = \wedge\{1 - \alpha : \|x_0\|_\alpha^* \leq t_0\} = 1$. Thus, $\nu(x_0, t_0) = \nu'(x_0, t_0) = 1$.

Case 4: $x_0 \neq \theta$, $t_0(> 0) \in R$ such that $\nu(x_0, t_0) = 0$. From (ix) it follows that $\|x_0\|_\alpha^* < t_0$, $\forall \alpha \in (0, 1)$. Therefore, $\|x_0\|_\alpha^* < t_0 \Rightarrow \nu'(x_0, t_0) = 0$, by (x). Thus, $\nu(x_0, t_0) = \nu'(x_0, t_0) = 0$.

Case 5: $x_0 \neq \theta$, $t_0(> 0) \in R$ such that $0 < \nu(x_0, t_0) < 1$. Let $\nu(x_0, t_0) = 1 - \beta$, then from (ix) we have

$$\|x\|_\alpha^* \leq t_0. \tag{5}$$

Using (5) from (x) we get, $\nu'(x_0, t_0) \leq 1 - \beta$. Therefore,

$$\nu(x_0, t_0) \geq \nu'(x_0, t_0). \tag{6}$$

Now, from Lemma (3.1.25) we have $\nu(x_0, t_0) = 1 - \beta \Leftrightarrow \|x\|_\alpha^* = t_0$. Now, for $\beta < \alpha < 1$, let $\|x\|_\alpha^* = t'$. Then again by Lemma (3.1.25), we have $\nu(x_0, t') = 1 - \alpha$. So, $\nu(x_0, t') = 1 - \alpha < 1 - \beta = \nu(x_0, t_0)$. Since $\nu(x_0, \cdot)$ is strictly monotonically decreasing and $\nu(x_0, t') < \nu(x_0, t_0)$ therefore $t' > t_0$. Then for $\beta < \alpha < 1$, we have $\|x\|_\alpha^* = t' > t_0$. So,

$$\nu'(x_0, t_0) \geq 1 - \beta = \nu(x_0, t_0). \tag{7}$$

Thus, from (6) and (7) we have $\nu(x_0, t_0) = \nu'(x_0, t_0)$. Since $(x_0, t_0) \in V \times R$ is arbitrary, $\nu'(x, t) = \nu(x, t_0)$ for all $(x, t) \in V \times R$.

Lemma (3.1.25)[6] In a fuzzy antinormed linear space (V,A^*) with respect to a t-conorm $\diamond$ satisfying (vi), (vii) and (viii), every sequence is convergent if and only if it is convergent with respect to its corresponding α-norms, $\alpha \in (0, 1)$.

Proof. ⇒ Part: Let (V,A^*) be a fuzzy antinormed linear space satisfying (vi) and (vii) and $\{x_n\}_{n\in N}$ be a sequence in V such that $x_n \to x$ $\lim_{n\to\infty} \nu(x_n - x, t) = 0, \forall t > 0$.

Choose $0 < \alpha < 1$. So, $\lim_{n\to\infty} \nu(x_n - x, t) = 0 < 1 - \alpha \Rightarrow$ there exists $n_o(t)$ such that

$$\nu(x_n - x, t) < 1 - \alpha, \forall n \geq n_0(t, \alpha). \quad (8)$$

Now,

$$\|x_n - x\|_\alpha^* = \wedge\{t > 0 : \nu(x_n - x, t) \leq 1 - \alpha\}$$

$$\Rightarrow \|x_n - x\|_\alpha^* \leq t, \forall n \geq n_0(t, \alpha).$$

Since $t > 0$ is arbitrary,

$$\|x_n - x\|_\alpha^* \to 0 \text{ as } n \to \infty, \forall \alpha \in (0, 1).$$

⇐ Part: Next we suppose that, $\|x_n - x\|_\alpha^* \to 0$ as $n \to \infty$, $\forall \alpha \in (0, 1)$.

Then for $\alpha \in (0, 1)$, $\varepsilon > 0$ there exists $n0(\alpha, ϙ)$ such that

$$\|x_n - x\|_\alpha^* < \varepsilon, \forall n \geq n_0(\alpha, \varepsilon), \alpha \in (0, 1). \quad (9)$$

Now,

$$\nu(x_n - x, \varepsilon) = \wedge\{1 - \alpha : \|x_n - x\|_\alpha^* \leq \varepsilon\}$$

$$\Rightarrow \nu(x_n - x, \varepsilon) \leq 1 - \alpha, \forall n \geq n_0(\alpha, \varepsilon), \alpha \in (0, 1)$$

$$\Rightarrow \lim_{n\to\infty} \nu(x_n - x, \varepsilon) = 0.$$

Thus x_n converges to x.

Corollary(3.1.26)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄ satisfying (vi), (vii) and (viii). $W(\subseteq V)$ is closed in (V,A^*) if and only if it is closed with respect to its corresponding α-norms, $\alpha \in (0, 1)$.

In the following lemma, a finite dimensional space is characterized by compact set in fuzzy environment and this will lead us to one of the fundamental differences between finite dimensional and infinite dimensional normed spaces with respect to fuzzy antinorms.

Lemma(3.1.27)[6] (Riesz) Let W be a closed and proper subspace of a fuzzy antinormed linear space (V, ν) with respect to a t-conorm ⋄, satisfying (vi), (vii) and (viii). Then for each $\varepsilon > 0$ there exists $y \in V - W$ such that $\nu(y, 1) \leq 1-\alpha$ and $\nu(y-w, 1-\varepsilon) \leq 1-\alpha$ for all $\alpha \in (0, 1)$ and $w \in W$.

Proof. Recall that, $\|x\|_\alpha^* = \wedge\{t : \nu(x, t) \leq 1 - \alpha\}$, $\alpha \in (0, 1)$ and $\{\|\cdot\|_\alpha^* : \alpha \in (0, 1)\}$ is an increasing family of α-norms on a linear space V . Now, by applying Riesz lemma for normed linear space, it follows that for any $\varepsilon > 0$ there exists $y \in V - W$ such that

$$\|y\|_\alpha^* = 1, \tag{10}$$

$$\|y-w\|_\alpha^* > 1 - \varepsilon, \ \forall w \in W. \tag{11}$$

Now, from Theorem (3.1.24) for all $\alpha \in (0, 1)$ we have

$$\nu(y, t) = \wedge\{1 - \alpha : \|y\|_\alpha^* \leq t\}$$

$$\Rightarrow \nu(y, 1) = \wedge\{1 - \alpha : \|y\|_\alpha^* \leq 1\}$$

$$\Rightarrow \nu(y, 1) \leq 1 - \alpha.$$

Again,

$$\nu(y - w, t) = \wedge\{1 - \alpha : \|y-w\|_\alpha^* \leq t\}$$

$$\Rightarrow \nu(y - w, \varepsilon) = \wedge\{1 - \alpha : \|y-w\|_\alpha^* \leq \varepsilon\}$$

$$\Rightarrow \nu(y - w, \varepsilon) \leq 1 - \alpha.$$

Hence the proof.

Theorem(3.1.28)[6] Let (V,A*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄, satisfying (vi), (vii) and (viii). If the set $\{x : \nu(x, 1) \le 1 - \alpha\}$, $\alpha \in (0, 1)$ is compact, then V is a space of finite dimension.

Proof. It can be easily verified that $\{x : \nu(x, 1) \le 1 - \alpha\} = \{x : \|x\|_\alpha^* \le 1\}$, $\alpha \in (0, 1)$. By applying Lemma (3.1.27), it can be proved that if for some $\alpha \in (0, 1)$ the set $\{x : \|x\|_\alpha^* \le 1\}$ is compact, then V is of finite dimensional. Using Lemma(3.1.28), it follows that, for some $\alpha \in (0, 1)$, $\{x : \nu(x, 1) \le 1-\alpha\}$ is compact, then V is a space of finite dimension.

Section (3-2) Fuzzy α-Anti-Convergence

The relations of fuzzy α-anti-convergence, fuzzy α-anti-Cauchyness, fuzzy α-anti-compactness with respect to their corresponding increasing family norms are studied.

Theorem(3.2.1)[6] Let (V,A*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄, satisfying (vi), (vii), (viii) and $\{\|\cdot\|_\alpha^* : \alpha \in (0, 1)\}$ be increasing family of norms of V, defined by (ix). Then, for any increasing (or, decreasing) sequence $\{\alpha n\}n \in N$ in (0, 1), $\alpha_n \to \alpha$ in (0, 1) implies $\|x\|_\alpha^*$, $\forall x \in V$.

Proof For $x = \theta$, it is clear that αn converges to $\alpha \Rightarrow \|x\|_{\alpha_n}^* \to \|x\|_\alpha^*$.

Suppose $x \ne \theta$. Then, from Lemma(3.1.28), for $x \ne \theta$, $\alpha \in (0, 1)$ and $t' > 0$, we have

$$\|x\|_\alpha^* = t' \Leftrightarrow \nu(x_0, t') = 1 - \alpha.$$

Let $\{\alpha_n\}_{n\in N}$ be an increasing sequence in (0, 1), such that αn converges to α in (0, 1). Let $\|x\|_{\alpha_n}^* = s_n$ and $\|x\|_\alpha^* = s$. Then,

$$\nu(x, s_{n)} = 1 - \alpha n \text{ and } \nu(x, s) = 1 - \alpha. \qquad (12)$$

Since $\{\|\cdot\|_{\alpha}^{*} : \alpha \in (0, 1)\}$ is an increasing family of norms, $\{s_n\}_{n\in N}$ is an increasing sequence of real numbers. Since $\{s_n\}_{n\in N}$ is an increasing sequence of real numbers and is bounded above by s, $\{s_n\}_{n\in N}$ is convergent. Thus,

$$\lim_{n\to\infty} \nu(x, s_n) = 1 - \lim_{n\to\infty} \alpha_n \Rightarrow \nu(x, s_n) = 1 - \alpha. \qquad (13)$$

From (12) and (13) we have $\nu(x, \lim_{n\to\infty} s_n) = \nu(x, s)$. This implies $\lim_{n\to\infty} s_n = s$, by (vii). Therefore, $\lim_{n\to\infty} \|x\|_{\alpha_n}^{*} = \|x\|_{\alpha}^{*}$.

Similarly, if $\{\alpha_n\}_{n\in N}$ is a decreasing sequence in (0, 1) and αn converges to α in (0, 1) then, it can be easily shown that $\|x\|_{\alpha_n}^{*} \to \|x\|_{\alpha}^{*}$, $\forall x \in V$.

Definition (3.2.2)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄ and $\alpha \in (0, 1)$. A sequence $\{x_n\}_{n\in N}$ in V is said to be fuzzy α-anti-convergent in (V,A^*), if there exists $x \in V$ such that for all $t > 0$

$$\lim_{n\to\infty} \nu(x_n - x, t) < 1 - \alpha.$$

Then x is called fuzzy α-antilimit of x_n.

Theorem (3.2.3)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄ satisfying (vi) and (viii). Then fuzzy α-antilimit of a fuzzy α-anti-convergent sequence is unique.

Proof Let $\{x_n\}_{n\in N}$ be a fuzzy α-anti-convergent sequence and suppose it converges to x and y in V . Then for all $t > 0$

$$\lim_{n\to\infty} \nu(x_n - x, t) < 1 - \alpha \text{ and } \lim_{n\to\infty} \nu(x_n - y, t) < 1 - \alpha.$$

Now,

$$\nu(x - y, t) = \nu(x - x_n + x_n - y, t), \forall n$$

$$= \nu(xn - x, t) \diamond \nu(x_n - y, t), \forall n.$$

Taking limit we have

$$\nu(x - y, t) = \lim_{n\to\infty} \nu(x_n - x, t) \diamond \lim_{n\to\infty} \nu(x_n - y, t)$$

$$< (1 - \alpha) \diamond (1 - \alpha) = (1 - \alpha), \text{ (by (viii))}.$$

That is, $\nu(x - y, t) < 1, \forall t > 0$. Therefore, $x - y = \theta$ by (vi) $\Rightarrow x = y$.

Theorem (3.2.4)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying (vi) and (viii). If $\{x_n\}_{n\in N}$ is a fuzzy α-anticonvergent sequence in (V,A^*) such that x_n converges to x, then $\|x_n - x\|_\alpha^* \to 0$ as $n \to \infty$.

Proof Since $\{x_n\}_{n\in N}$ be a fuzzy α-anti-convergent sequence, suppose it converges to x, then for all $t > 0$, $\lim_{n\to\infty} \nu(x_n - x, t) < 1 - \alpha$

$$\Rightarrow \exists n_0(t) > 0 \text{ such that } \nu(x_n - x, t) < 1 - \alpha, \forall n \geq n_0(t)$$

$$\Rightarrow \exists n_0(t) > 0 \text{ such that } \|x_n - x\|_\alpha^* \leq t, \forall n \geq n_0(t).$$

Since $t > 0$ is arbitrary, $\|x_n - x\|_\alpha^* \to 0$ as $n \to \infty$.

Definition(3.2.5)[6] Let (V,A^*) be a fuzzy anti-normed linear space with respect to a t-conorm $\diamond$ and $\alpha \in (0, 1)$. A sequence $\{x_n\}_{n\in N}$ in V is said to be fuzzy α-anti-Cauchy sequence if

$$\lim_{n\to\infty} \nu(x_n - x_{n+p}, t) \leq 1 - \alpha, \forall t > 0, p = 1, 2, 3, \ldots$$

Theorem (3.2.7)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying (viii) and $\alpha \in (0, 1)$. Then every fuzzy α-anticonvergent sequence in (V,A^*) is a fuzzy α-anti-Cauchy sequence in (V,A^*).

Proof. Let $\{x_n\}_{n\in N}$ be a fuzzy α-anti-convergent sequence and it converging to x. Then

$$\lim_{n\to\infty} \nu(x_n - x, t) < 1 - \alpha.$$

Now,

$$\nu(x_n - x_{n+p}, t) = \nu(x_n - x + x - x_{n+p}, t), \text{ for } p = 1, 2, 3, \ldots$$

$$= \nu\left(x_n - x, \frac{t}{2}\right) \Diamond \nu\left(x_{n+p} - x, \frac{t}{2}\right) \text{ for } p = 1, 2, 3, \ldots$$

Therefore,

$$\lim_{n\to\infty} \nu(x_n - x_{n+p}, t) \leq \lim_{n\to\infty} \nu\left(x_n - x, \frac{t}{2}\right) \diamond \lim_{n\to\infty} \nu\left(x_{n+p} - x, \frac{t}{2}\right)$$

$$< (1 - \alpha) \diamond (1 - \alpha) = (1 - \alpha), \text{ (by (viii))}.$$

Hence, $\{x_n\}_{n\in N}$ is a fuzzy α-anti-Cauchy sequence in (V,A^*).

Theorem (3.2.8)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm ⋄, satisfying (vi) and (viii). Then every Cauchy sequence in $(V, \|\cdot\|_\alpha^*)$ is a fuzzy α-anti-Cauchy sequence in (V,A^*), where $\|\cdot\|_\alpha^*$ denotes the increasing family of norms on V defined by (ix), $\alpha \in (0, 1)$.

Proof. Choose α0 ∈ (0, 1) arbitrary but fixed. Let $\{y_n\}_{n\in N}$ be a Cauchy sequence in V with respect to $\|\cdot\|_{\alpha_0}^*$. Then

$$\lim_{n\to\infty} \|y_n - y_{n+p}\|_{\alpha_0}^* = 0.$$

Then for any given ε (> 0) there exists a positive integer $n_0(\varepsilon)$ such that $\|y_n - y_{n+p}\|_{\alpha_0}^* < \varepsilon$, $\forall n \geq n_0(\varepsilon)$ and p = 1, 2, 3, ...

$$\Rightarrow \wedge\{t > 0 : \nu(y_n - y_{n+p}, t) \leq 1 - \alpha_0\} < \varepsilon,$$

⇒ there exists t(n, p, ε) < ε such that

$$\nu(y_n - y_{n+p}, t(n, p, \varepsilon)) \leq 1 - \alpha_0, \forall n \geq n_0(\varepsilon) \text{ and } p = 1, 2, 3, \ldots$$

$$\Rightarrow \nu(y_n - y_{n+p}, \varepsilon) \leq 1 - \alpha_0.$$

Since ε (> 0) is arbitrary,

$$\lim_{n\to\infty} \nu(y_n - y_{n+p}, t) \leq 1 - \alpha_0, \forall t > 0 \Rightarrow \{y_n\}_{n\in N}$$

is fuzzy α_0-anti-Cauchy sequence in (V,A^*).

Since $\alpha_0 \in (0, 1)$ is arbitrary, every Cauchy sequence in $(V, \|\cdot\|_\alpha^*)$ is fuzzy α-anti-Cauchy sequence in (V,A^*) for each $\alpha \in (0, 1)$.

Definition (3.2.9)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$ and $\alpha \in (0, 1)$. It is said to be fuzzy α-anti-complete if every fuzzy α-anti-Cauchy sequence in V fuzzy α-anti-converges to a point of V .

Theorem (3.2.10)[6] Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying (vi) and (viii). If (V,A^*) is fuzzy α-anti-complete then V is complete with respect to $\|\cdot\|_\alpha^*$, $\alpha \in (0, 1)$.

Proof Choose $\alpha_0 \in (0, 1)$ arbitrary but fixed. Let $\{y_n\}_{n\in N}$ be a Cauchy sequence in V with respect to $\|\cdot\|_\alpha^*$, then $\{y_n\}_{n\in N}$ is fuzzy α_0-anti-Cauchy sequence in (V,A^*). Since (V,A^*) is fuzzy α_0-anti-complete, there exists $y \in V$ such that

$\lim_{n\to\infty} \nu(y_n - y, t) < 1 - \alpha_0, \forall t > 0 \Rightarrow \lim_{n\to\infty} \|y_n - y\|_{\alpha_0}^* = 0$, by Theorem (3.2.8).

$\Rightarrow y_n \to y$ with respect to $\|\cdot\|_{\alpha_0}^*$.

$\Rightarrow (V, \|\cdot\|_{\alpha_0}^*)$ is complete.

Since α_0 is arbitrary, $(V, \|\cdot\|_{\alpha_0}^*)$ is complete.

Section (3-3) Some Topological and Algebraic Properties of α-level Subsets' Topology of a Fuzzy Subset

Zadeh introduced the general theory of fuzzy sets. He laid the foundation for the concept of complement, union, intersection and emptiness of fuzzy set. Chang was the first person to extend these concepts to a topological space. He defined a fuzzy topology on a set X as the collection of all its fuzzy subsets satisfying three axioms similar to those of a classical fuzzy topological space.

Wong considered some properties of Chang's fuzzy topological space such as cover, sequential compactness and semi compactness and concluded that the theory of fuzzy topological spaces in these regards are less useful. He also developed a product and quotient fuzzy topology on Chang's fuzzy topological space. He examined some properties of fuzzy homeomorphism which he regarded as an F-continuous one-to-one mapping between two fuzzy topological spaces such that the inverse also is F-continuous. He further introduced the concept of fuzzy point in order to formulate the idea of fuzzy convergence.

However, noting that, according to Chang's fuzzy topological space, constant maps between fuzzy topological spaces are not continuous, Lowen modified the definition of fuzzy topology on X by Chang. Hence, he referred to Chang's topology as a quasi topology. Along side, he formulated and extended some ideas such as continuity and compactness. But Gantner faulted the work of Lowen as losing the concept of generalization which fuzzy topology on X is to yield. Sarka considered some Hausdorff separation axioms by taking the fuzzy elements into consideration and, with this, he hoped to improve on the work of Gantner which he considered to have done a similar thing but for only the crisp points and crisp subsets of fuzzy spaces. Shostak made a modification that is completely different from the existing concept of fuzzy topology by introducing that a fuzzy set can be open or closed to a degree. Hence a fuzzy topological function

$$\tau(\alpha) = \begin{cases} 1, & \text{if } \alpha \text{ is complete open} \\ \beta, & \text{if } \alpha \text{ is open to a degree } \beta \in (0,1) \\ 0, & \text{if } \alpha \text{ is complete not open.} \end{cases}$$

Chakrabarty et al took a completely different line of thought by introducing fuzzy topology on a fuzzy set rather than on X, using a tolerance relation $\overline{S}$. This was considered tolerance topology and the pair ($\overline{A}$, $\overline{S}$) a tolerance space, where $\overline{A}$ is a fuzzy set. Chaudhuri et al built on this topology to develop the concepts such as Hausdorffness, regularity, normality and completeness of normality. Das continued in this setting and introduced a product topology and fuzzy topological group. However, there was a resurgence of gradation of openness of Shostak. This, according to Gregori et al, makes it easier to avoid the concept of fuzzy point. The concepts of a-level openness, α-interior and a-neighbourhood were also introduced. This idea was also followed by Benchalli to obtain some fuzzy topological properties such as a-Hausdorffness, a-connectedness and α-compactness. Onasanya also introduced and studied some properties such as fuzzy accumulation (or cluster) points of an α-level subset of a fuzzy topological space instead of that of fuzzy cluster set introduced by Chang. This also makes it easier to avoid fuzzy points. We now introduce a setting where the topology we introduce is on the fuzzy set m itself and examine some properties of this topology. This is not a tolerance topology as in Chakrabarty et al because it uses the collection of level subsets. It is also different from the topology by a collection of mere fuzzy subsets as in Chang. It is also different from the topology in Shostak because this topology is not on X.

Definition (3.3.1)[4] Let X be a non-empty set. The fuzzy subset m of the set X is a function μ: X $\longrightarrow$ [0,1], where μ_X is the membership function of the fuzzy set μ. We can just use m for μ_X since it is characteristic of the fuzzy set μ.

Definition(3.3.2)[4] Let μ and λ be any two fuzzy subsets of a set X. Then

(i) λ and m are equal if $\mu(x) = \lambda(x)$ for every x in X

(ii) λ and μ are disjoint if $\mu(x) = \lambda(x)$ for every x in X

(iii) $\lambda \subseteq \mu$ if $\mu(x) > \lambda(x)$

Definition(3.3.3)[4] Let μ be the fuzzy subset of X. Then, for some $\alpha \in [0,1]$, the set $\mu_\alpha = \{x \in X : \mu(x) \geq \alpha\}$ is called a α-level subset of the fuzzy subset μ. If $\alpha_1 \leq \alpha_2$, then $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$.

Definition (3.3.4)[4] Let μ be a fuzzy subset of X. Then, λ is the complement of m if $\lambda(x) = 1 - \mu(x)$ $\forall x \in X$.

Definition (3.3.5)[4] The family $T = \{A_j\}_{j \in I}$ of fuzzy subsets of X such that

(i) Φ, X $\in$ T,

(ii) $\cup_j A_j \in T$ for each $j \in I$ and

(iii) $A_k \cap A_j \in T$ for each k, $i \in I$

is called a fuzzy topology on X and (X, T) is called a fuzzy topological space.

Remark (3.3.6)[4] $\Phi = \{x \in X : \mu(x) = 0\ \forall x \in X\}$ and $X = \{x \in X : \mu(x) = 1\ \forall x \in X\}$. Every member of T is called T-open or simply open fuzzy set. Alternative to this is the definition where κ is the fuzzy set in X with constant value λ. Then, $\kappa_0 = \Phi$ and $\kappa_1 = X$.

Definition(3.3.7)[4] A fuzzy set U in a fuzzy topological space is a neighbourhood of a fuzzy set m if there exists an open fuzzy set O such that $\mu \subset O \subset U$.

Remark (3.3.8)[4] The collection of all such neighborhood U of μ is called neighborhood system of μ. In this case, μ is called the interior fuzzy set of U and the collection of all the interior fuzzy sets is called the interior of U and can be denoted U°. It is important to note that a fuzzy set m is open if and only if $\mu = \mu°$.

Theorem(3.3.9)[4] A fuzzy set is open if and only if it is the neighbourhood, of each fuzzy set in it.

Definition (3.3.10)[4] A sequence of fuzzy sets, say $\{A_n : n = 1, 2, 3, ...\}$, is eventually contained in a fuzzy set A if and only if there is an integer m such that, if $n \geq m$, then $A_n \subset A$.

Definition (3.3.11)[4] A sequence of fuzzy sets, say $\{A_n : n = 1, 2, 3, ...\}$, is frequently contained in a fuzzy set A if and only if for each integer m there is an integer n such that, if $n \geq m$, then $A_n \in A$.

Remark (3.3.12)[4] A sequence of fuzzy sets $\{A_n : n = 1, 2, 3, ...\}$ in a fuzzy topological space is said to converge to a fuzzy set A if and only if it is eventually contained in each neighborhood of A.

Definition (3.3.13)[4] A fuzzy set A in a fuzzy topological space is a cluster fuzzy set of a sequence of fuzzy sets if and only if the sequence is frequently contained in every neighbourhood of A.

Definition (3.3.14)[4] A fuzzy point x_p in a space of points say, X, is a fuzzy set μ_{x_p} defined by

$$\mu_{x_p}(x) = \begin{cases} \neq 0, & \text{when } x = x_p \\ 0, & \text{elsewise.} \end{cases}$$

The implication of this is that when $\mu_{x_p}(x)$ is restricted to $X - \{x_p\}$ it is an improper (or constant) fuzzy set with membership value 0.

Theorem (3.3.15)[4] Let A_α be a α-level subset of a fuzzy topological space (X, T). A point $x \in X$ is a fuzzy accumulation point or fuzzy cluster point of A_α if for T-open subset U containing x it is such that $U \subset U_\beta$ with $A_\alpha \cap U \setminus \{x\} \neq 0$ and $\beta < \alpha$.

Definition(3.3.16)[4] A class $\{A_i\}$ of sets is said to have finite intersection property if every finite subclass has a non-empty intersection.

Definition(3.3.17)[4] A fuzzy topological space is sequentially compact if every sequence in it converges to a point in it.

Main Results(3.3.18)[4]

We now introduce a new topology τ^* which is defined on a fuzzy subset μ of a non-empty set X rather than on X itself. The topology is the collection of α-level subsets $\{\mu_{\alpha_i}\}$ of X.

In what follows, we shall show that the collection of α-level subsets of X defines a topology on the fuzzy subset μ of X. Some basic and useful definitions shall be given. Some of these look like Chang's definition but are modified to fit into the new topology.

Theorem(3.3.19)[4] The family $\tau^* = \{\mu_{\alpha_i} | \mu(x) > \alpha_i\} \setminus X$ for $\alpha_i \in [0, 1]$ defines a topology on μ.

Proof. Note that we have dropped the level subset $X = \mu_{\alpha_m}$, for some $\alpha_m = 1$ in which case $\mu(x)=1\ \forall x \in X$. Then,

$\mu_{\alpha_i} = \{x \in X : \mu(x) > \alpha_t \in [0,1]\} = \mu$ and $\mu_{\alpha_n} = \{x \in X : \mu(x) = 0\ \forall x \in X$ for some $\alpha_n = 0\} = \mu_0$. Hence, $\mu, \mu_0 \in \tau^*$.

If we also consider that $\mu_{\alpha_1}, \mu_{\alpha_2} \in \tau^*$ and take $0 = \max\{\alpha_1, \alpha_2\}$.,

$$\mu_{\alpha_1} \cap \mu_{\alpha_2} = \mu_\beta \in \tau^*.$$

Further, for $\{\mu_{\alpha_1}\}_{i=1}^{\infty}$ such that for every i, $\mu_{\alpha_1} \in \tau^*$. Let $\gamma = \min\{\alpha_i\}$. By, $\bigcup_{i=1}^{\infty} \mu_{\alpha_1} = \mu_\gamma \in \tau^*$.

Remark (3.3.20)[4] It should be noted that this family has a noetherian property. In this regard, (μ, τ^*) is a topological space and this we refer to as fuzzy level topological space or τ^*-space. Every element $\mu_\alpha \in \tau^*$ is called level open or τ^*-open. Hence, a level set μ_β is τ^*-closed if and only if its complement μ_β' is τ^*-open. This level openness agrees to the definition of level openness . A fuzzy set λ can be said to be open if it coincides with a level subset of μ. When $\tau^* = \{\mu, \mu_0\}$ we have an indiscrete level topological space. If $\tau^* = \{\mu, \mu_0, \mu_{\alpha_1}, \mu_{\alpha_2}, \mu_{\alpha_3}, \ldots, \mu_{\alpha_n}\}$, where $\{\mu_{\alpha_i}\}$ is a collection of all possible level subsets of μ, we have a discrete level topological space.

Definition(3.3.21)[4] The sequence of level subsets $\{\mu_{\alpha_i}\}$ in τ^* is frequently contained in a level subset μ_{α_k} if for each i there is an i_0 such that for $\alpha_{i_0} \geq \alpha_i$ we have $\mu_{\alpha_i} \subseteq \mu_{\alpha_k}$.

Definition (3.3.22)[4] The sequence of level subsets $\{\mu_{\alpha_i}\}$ is eventually contained in a level subset μ_{α_k} if there is an i_0 such that for $\alpha_i \geq \alpha_{i_0}$ we have $\mu_{\alpha_i} \subseteq \mu_{\alpha_k}$.

Definition(3.3.23)[4] The sequence of level subsets $\{\mu_{\alpha i}\}$ is said to converge to a level subset μ_m if the sequence is eventually contained in every neighbourhood of μ_m.

Definition(3.3.24)[4] A level subset $\mu_{\alpha 0}$in the sequence $\{\mu_{\alpha_i}\}$ is called maximal if $\mu_{\alpha_i} \subseteq \alpha_{i_0}$ for all i such that $\alpha_0 \leq \alpha_i$ or $\alpha = \min\{\alpha_i\}$ and it is minimal if $\alpha_{i_0} \subseteq \mu_{\alpha_i}$ for all i such that $\alpha_0 \geq \alpha_i$or $\alpha_0 = \max\{\alpha_i\}$.

Remark (3.3.25)[4] It is can be observed that the sequence is frequently contained in the maximal level subset. This is because, for any α_i there is an α_k such that $\alpha_i \geq \alpha_k \geq \alpha_0$, then $\mu_{\alpha_i} \subseteq \mu_{\alpha_0}$.

Definition(3.3.26)[4] The sequence $\{\mu_{\alpha_i}\}$ of fuzzy level subsets is bounded if it has both the minimal and maximal level subsets.

Definition (3.3.27)[4] A fuzzy level subset $\mu_{\alpha 0}$ in (μ, τ^*) is a cluster fuzzy level subset of a sequence of fuzzy level subset $\{\mu_{\alpha_i}\}$ if the sequence is frequently contained in every neighbourhood of α_{i_0}.

Definition(3.3.28)[4] A fuzzy subset λ in (μ, τ^*) is a neighbourhood of a level subset μ_{α_k} if there is a level open set O_{α_m} such that $\mu_{\alpha_k} \subset O_{\alpha_m} \subset \lambda$.

Remark (3.3.29)[4] Though it cannot be said that an element x is in μ, x except only to a degree, but rather it can be said that $x \in \mu_\alpha$. Hence, we can define a fuzzy subset μ as a neighbourhood of an element x if there is a τ^*-open level subset O_α such that $x \in O_\alpha \in \mu$. Definition (3.3.28) is a generalisation of this since if μ is a neighbourhood of any μ_α, it is a neighbourhood of each point of μ_α.

Definition (3.3.30)[4] The level subset μ_{α_k} is called an interior fuzzy level subset of the fuzzy set A and each point of μ_{α_k} is called fuzzy interior point of λ. The collection of all such μ_{α_k} is the interior of λ.

Definition (3.3.31)[4] The collection of all such neighbourhoods λ of level subset (or of points of) μ_{α_k} is called the neighbourhood system (or of the points) of μ_{α_k} and is denoted N.

Theorem (3.3.32)[4] A fuzzy subset A in a fuzzy topological space (μ, τ^*) is τ^*-open if and only if λ is a neighbourhood of every μ_{α_i} in it.

Proof. Let λ be open in M and $\mu_{\alpha_i} \subset \lambda$ for each i. Then, λ being opened is in τ^*, there is a $\beta \in R$ such that $\mu_{\alpha_i} \subset \lambda = \mu_\beta$ with $\beta < \alpha_i$. Since β and α_i are distinct real numbers, (β, α_i) forms an interval. So there is a $\gamma \in R$ such that $\beta < \gamma < \alpha_i$ which implies that $\mu_{\alpha_i} \subset \mu_\gamma \subset \mu_\beta$. But $\mu_\gamma \in \tau^*$. Hence, for every $\mu_{\alpha_i} \subset \lambda$, there is $\mu_\gamma \in \tau^*$ such that

$$\mu_{\alpha_i} \subset \mu_\gamma \subset \lambda = \mu_\beta .$$

Thus, λ is a neighbourhood of all μ_{α_i}'s in it.

Conversely, let λ be a neighbourhood of each μ_{α_i} in it. Then, $U\mu_{\alpha i} \subseteq \lambda$. But each $\mu_{\alpha_i} \subset \lambda$ for each i so that for each $x \in \mu_{\alpha_i} \subset \lambda$, $\mu(X) > \alpha_i$. But each x which is to a certain degree contained in λ is in some $\mu_{\alpha i}$ so that $\lambda \subseteq U\mu_{\alpha_i}$. Then, $\lambda = U\mu_{\alpha_i} \in \tau^*$. Thus, λ is open.

The following theorem connects Chang's topological space with τ^*-space.

Theorem (3.3.33)[4] Let the fuzzy set λ be a neighbourhood of the fuzzy set μ in Chang's topological space. Then, λ is a neighbourhood of all the τ^*-open subsets μ_α of μ.

Proof. Note that $\mu_\alpha \subset \mu$ for every a. Since λ is a neighbourhood of μ, there is an open set O in Chang's topological space such that, $\mu \subset O_\alpha \subset A$.

But for every α, $\mu_\alpha \subset \mu \subset O \subset \lambda$. Hence, $\mu_\alpha \subset O \subset \lambda$.

Theorem (3.3.34)[4] λ sequence of fuzzy subsets that is frequently contained in a fuzzy subset M is eventually contained in μ.

Proof. For each m such that $n \geq m$, we have $A_n \subset \lambda$. But we can choose a fix m_0 such that for $n \geq m_0$, $A_n \subset \lambda$. Hence, $\{A_n\}$ is eventually contained in λ.

Theorem (3.3.35)[4]Every sequence of level subsets $\{\mu_{\alpha_i}\}$ of fuzzy set M is frequently contained in the fuzzy set μ.

Proof. For each m such that $\alpha_i \geq m$, $\mu_{\alpha_i} \subseteq \mu_m$. Since for each m, $\mu_{\alpha_i} \subseteq \mu_m \subset \mu$, we have $\mu_{\alpha_i} \subseteq \mu$, the sequence { } is frequently contained in μ.

Remark(3.3.36)[4] The sequence$\{\mu_{\alpha_i}\}$is eventually contained in μ.

Theorem (3.3.37)[4] Every level subset of μ in the sequence $\{\mu_{\alpha_i}\}$, except the maximal one, is a fuzzy cluster level subset.

Proof. Let μ_{α_0} be any fuzzy level subset of μ and μ_α the neighbourhood of μ_{α_0}. Always, $\mu_{\alpha_0} \subset \mu_\alpha$ with $\alpha_0 \geq \alpha$. For each a such that $\alpha_o \geq \alpha$, $\mu_{\alpha_i} \subseteq \mu_\alpha$ and so the sequence is frequently contained in μ_α. As for the maximal one, it is not properly contained in any other level subset which can serve as its neighbourhood.

Corollary (3.3.38)[4] The sequence $\{\mu_{\alpha_i}\}$ converges to its minimal fuzzy level cluster subset μ_{α_i}.

Proof. The sequence converges to its minimal level subset .And it is one of the fuzzy cluster level subset .

Theorem(3.3.39)[4] The fuzzy set m is open if and only if it can be expressed as the union of all its fuzzy cluster level subsets.

Proof. If $\mu = U_{\alpha i}\ \mu_{\alpha_i} = \max\{\mu_{\alpha_i}\}$. Then there is an $\alpha_k = \min\{\alpha_i\}$ such that $\mu = \mu_{\alpha_k} \in \tau^*$. Thus, m is open.

Conversely, if m is open, we have $\mu \in \tau^*$. Then, there is an α_k such that $\mu_{\alpha_k} = \mu$.

Remark(3.3.40)[4] The following is similar to what we have in classical case that a set is closed if and only if it contains all its limit points, in which case the set is bounded.

Theorem(3.3.41)[4] The sequence of level subsets is bounded if and only if it contains all its cluster level subsets of the fuzzy set.

Proof. If a sequence contains all the cluster level subsets of the fuzzy set, they are also level subsets of that fuzzy set, the sequence is bounded. Conversely, if the sequence is bounded, it has both the minimal and maximal level subsets. Since it is a noetherian sequence, it contains the maximal level subset, and thus the sequence, contains all the other level subsets of the fuzzy set. But all of them are cluster level subsets,except the maximal one.

Definition(3.3.42)[4]. The sequence of level subsets $\{\mu_{\alpha_i}\}$ is a cover for μ_{α_0} if $\mu_{\alpha_0} \subset \mathrm{U}\, \mu_{\alpha_i}$.

Remark(3.3.43)[4] The property of compactness of μ is hard to come by in this space because of the neotherian property of $\{\mu_{\alpha i}\}$. We rather can have something that mimics it. As a matter of fact, for the same reason, this space does not separate points. So, the property of Hausdorffness is not possible. But the space (μ, τ^*) has something that much resembles sequential compactness.

Definition(3.3.44)[4] The sequence of level subsets $\{\mu_{\alpha_i}\}$ is a quasicover for μ_{α_0} if $\mu_{\alpha_0} \subseteq \mathrm{U}\mu_{\alpha_i}$.

Definition(3.3.45)[4] A level subset μ_{α_0} is quasicompact if every open quasicover $\{\mu_{\alpha i}\}$ has a refinement or a subsequence $\{\mu_{\alpha_j}\}$ with $\alpha_j \geq \alpha_i$ which is a quasicover of μ_{α_0}.

Definition(3.3.46)[4] A fuzzy topological space is parasequentially compact if every sequence of level subsets in it converges to a level subset in it.

Theorem(3.3.47)[4] (μ, τ^*) is a parasequentially compact space.

Proof. Every sequence of level subsets in (μ, τ^*) converges to a level subset $\mu_{\alpha_0} \subseteq \mu$. Hence, (μ, τ^*) is parasequentially compact.

Chapter 4

Some Topological Properties of Fuzzy Antinormed Linear Spaces

Section(4-1) Some Topological Properties of Fuzzy Antinormed Linear Spaces

The concept of fuzzy set was introduced by Zadeh in 1965. Thereafter, fuzzy set theory found applications in different areas of mathematics and its applications in other sciences. The concept of fuzzy norm was introduced by Katsaras in 1984. In 1992, by using fuzzy numbers, Felbin introduced the fuzzy norm on a linear space. Cheng and Mordeson introduced another idea of fuzzy norm on a linear space, and in 2003 Bag and Samanta modified the definition of fuzzy norm of Cheng-Mordeson. A comparative study of the fuzzy norms defined by Katsaras, Felbin, and Bag and Samanta was given. The idea of fuzzy antinorm was introduced.

On the basis of this idea, Jebril and Samanta introduced the concept of fuzzy antinorm on a linear space based on the notion of continuous triangular conorm, first applied in investigation of probabilistic metric spaces. Dinda, Samanta, and Jebril further modified this concept and also defined fuzzy α-anticonvergence. We use this later approach to investigate statistical versions of anticonvergence. Recall that statistical convergence is defined by

using the asymptotic or natural density $\delta(A)$ of a subset A of natural numbers N, defined by $\delta(A) = \lim_{n\to\infty}(|\{k \in A : k \leq n\}|/n)$ if this limit exists. A⊂N is said to be statistically dense if $\delta(A)= 1$. consider somecovering properties in fuzzy antinormed linear spaces.

Preliminaries(4.1.1)[11] Contains some basic definition and preliminary results which we need for further exposition.

Definition (4.1.2)[11] Abinary operation $\Diamond : [0, 1]\times[0, 1]\rightarrow[0, 1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(i) $\Diamond$ is commutative and associative.

(ii) $\Diamond$ is continuous.

(iii) $a \Diamond 0 = a$, $a \in [0,1]$.

(iv) $a \le c$ and $b \le d$ imply $a \Diamond b \le c \Diamond d$ for $a, b, c, d \in [0, 1]$.

Classical examples of continuous t-conorms are

$a \Diamond b = a + b - ab$, $a \Diamond b = \max\{a, b\}$, $a \Diamond b = \min\{a + b, 1\}$.

We often use idempotent t-conorms $\Diamond$ (i.e., satisfying $a \Diamond b = a$ for each $a \in [0,1]$).

Recall now the notion of fuzzy antinorm in a linear space with respect to a continuous t-conorm following.

In what follows E will denote a real linear space with the zero element θ and $\Diamond$ will be a continuous t-conorm.

Definition(4.1.3)[11] Let E be a real linear space and $\Diamond$ a

t-conorm. A fuzzy subset $\nu : E \times R\rightarrow R$ of $E \times R$ is called a fuzzy antinorm on E with respect to the t-conorm $\Diamond$ if, for all $x, y \in E$,

(FaN1) for each $t \in (-\infty,0]$, $\nu(x, t) = 1$;

(FaN2) for each $t \in (0,\infty)$, $\nu(x, t) = 0$ if and only if $x = \theta$;

(FaN3) for each $t \in (0,\infty)$, $\nu(cx, t) = \nu(x, t/|c|)$ if $x \neq 0$;

(FaN4) for all $t, s \in R$, $\nu(x+y, s+t) \le \nu(x, s)\Diamond\nu(y, t)$;

(FaN5) $\lim_{n\to\infty} \nu(s, t) = 0$.

Note that if ν is the antinorm ν in the definition above, then ν(s, t) is nonincreasing with respect to t for each x ∈ E.

The following are examples of fuzzy antinorms with respect to a corresponding t-conorm and show how a fuzzy antinorm can be obtained from a norm.

Example(4.1.4)[11] Let (E, ||.||) be a normed linear space and let the t-norm ◊ be given by a ◊ b =a + b− ab. Define ν: E × R → [0, 1] by

$$\nu(x,t)=\begin{cases}0, & \text{if } t>\|x\|;\\ \frac{\|x\|}{t+\|x\|}, & \text{if } 0<t\le\|x\|,\\ 1, & \text{if } t\le 0.\end{cases} \tag{1}$$

Then ν is a fuzzy antinormon E with respect to the t-conorm ◊.

Example(4.1.5)[11] Let (E, || . ||)be a normed linear space and let the t- conorm ◊ be given by a ◊ b = max{a, b}. Define
ν: E × R→[0, 1] by

$$\nu(x,t)=\begin{cases}\frac{\|x\|}{t+\|x\|}, & \text{if } t>0;\\ 1, & \text{if } t\le 0.\end{cases} \tag{2}$$

Then ν is a fuzzy antinorm on E with respect to the t-conorm ◊.

Example(4.1.6)[11] Let (E, || . ||) be a normed linear space and let the t-conorm ◊ be given by a ◊ b = a + b−ab.

Define ν : E × R→ [0, 1] by

$$\nu(x,t)=\begin{cases}0, & \text{if } t>\|x\|;\\ 1, & \text{if } t\le\|x\|.\end{cases} \tag{3}$$

Then ν is a fuzzy antinormon E with respect to the t-conorm ◊. This antinorm ν satisfies also the following:

$$\nu(x, t) < 1 \text{ for each } t > 0 \text{ implies } x = \theta.$$

Example(4.1.7)[11] Let (E, ||.||) be a normed linear space and consider the t-conorm ◊ defined by a ◊ b = min{a + b, 1}.

Define ν: E×R→[0, 1] by

$$\nu(x,t)=\begin{cases}\frac{\|x\|}{2t-\|x\|}, & \text{if } t>\|x\|;\\ 1, & \text{if } t\le\|x\|.\end{cases} \tag{4}$$

Then ν is a fuzzy antinorm on E with respect to the t-norm ◊. Note that this ν satisfies the condition and also the following:

ν(x, .) is a continuous function on R and strictly decreasing on the subset {t : 0 < ν(x, t) < 1} of R.

Definition(4.1.8)[11] Asequence $(x_n)_{n\in N}$ in a fuzzy antinormed linear space (E, ν, ◊) is said to be ν-convergent to a point x ∈ E if for each $\varepsilon > 0$ and each t > 0 there is $n_0 \in N$ such that

$$\nu(x_n - x, t) < \varepsilon \text{ for each } n \ge n_0. \tag{5}$$

In this case we write $(x_n) \xrightarrow{\nu} x$.

Definition(4.1.9)[11] Asequence $(x_n)_{n\in N}$ in a fuzzy antinormed linear space (E, ν, ◊) is said to be statistically]-convergent to a point x∈E if for each $\varepsilon > 0$ and each t > 0

$$\delta(\{n \in N : \nu(x_n - x, t) < \varepsilon\}) = 0. \tag{6}$$

In this case we write $(x_n) \xrightarrow{st-\nu} x$.

Theorem(4.1.10)[11] Let (E, ν, ◊) be a fuzzy antinormed linear space with respect to an idempotent t-conorm ◊, and let ν satisfy (FaN6). Then for each $\lambda \in (0, 1)$ the function $||.||_\lambda : X \to [0,\infty)$ defined by

$$||.||_\lambda = \wedge\{t > 0 : \nu(x, t) \le 1 - \lambda\} \tag{7}$$

is a norm on E (called an λ-norm generated by ν), and $\Lambda = \{||.||_\lambda : \lambda \in (0,1)\}$ is an ascending family of norms on E.

Convention. We use the notation (E, Λ) for the family of normed linear spaces $\{(E, ||.||_\lambda) : \lambda \in (0, 1)\}$ and call (E, Λ) also a fuzzy antinormed linear space.

Lemma(2.1.11)[11] In a fuzzy antinormed linear space (E, Λ) with respect to an idempotent t-conorm ◊ satisfying (FaN6) and (FaN7) a sequence is statistically ν-convergent if and only if it is statistically λ-convergent for each $\lambda \in (0, 1)$.

Proof. (⇒): Let $(x_n)_{n\in N}$ be a sequence in E such that $(x_n)_{n\in N} \xrightarrow{st-\nu} x$, i.e., for each $t > 0$.

$$st - \lim_{n\to\infty} \nu\,(x_n - x, t) = 0. \tag{8}$$

Fix $\lambda \in (0, 1)$. So, $st - \lim_{n\to\infty} \nu\,(x_n - x, t) = 0 < 1 - \lambda$. There exists a statistically dense set $M \subset N$ so that, for each $n \in M$,

$$\nu\,(x_n - x, t) < 1 - \lambda. \tag{9}$$

Since $||x_n - x||_\lambda = \wedge\{t > 0 : \nu\,(x_n - x, t) \le 1 - \lambda\}$ we have $||x_n - x||_\lambda \le t$ for all $n \in M$. As $t > 0$ was arbitrary, for each $\lambda \in (0, 1)$, by (FaN6), we have $||x_n - x||_\lambda$ which statistically converges to 0.

(⇐): Suppose now that for each $\lambda \in (0, 1)$, $||x_n - x||_\lambda$ statistically converges to 0.Thismeans that for each $\lambda \in (0, 1)$ and each $\varepsilon > 0$ there is a statistically dense $M_\lambda \subset N$ such that

$$||x_n - x||_\lambda < \varepsilon \tag{10}$$

for each $n \in M_\lambda$.Therefore,

$$\nu\,(x_n - x, \varepsilon) = \wedge\{1 - \lambda : ||x_n - x||_\lambda \le \varepsilon\} \tag{11}$$

implies

$$\nu\,(x_n - x, \varepsilon) \le 1 - \lambda \tag{12}$$

for each $\lambda \in (0, 1)$ and each $n \in M_\lambda$, which means

$$\nu\,(x_n - x, \varepsilon) \xrightarrow{st-\nu} x; \tag{13}$$

$$\text{that is,} \quad (x_n) \xrightarrow{st-\nu} x. \tag{14}$$

The relations of fuzzy λ-anticonvergence, fuzzy λ- anticauchyness, and fuzzy λ-anticompactness with respect to their corresponding increasing family of norms are studied in the following part of this section.

Definition (4.1.12)[11] Let (E, Λ) be a fuzzy antinormed linear space and $\lambda \in (0, 1)$. A sequence $(x_n)_{n\in N}$ in E is said to be fuzzy statistically λ-anticonvergent in E if there existx $\in$ E and M $\subset$N with $\delta(M) = 1$ such that, for all t > 0,

$$\nu\,(x_n - x, \varepsilon) < 1 - \lambda. \tag{15}$$

In this case we write $(x_n) \xrightarrow{st-\nu} x$ and x is called a fuzzy statistical λ-antilimit of $(x_n)_{n\in N}$.

Theorem(4.1.13)[11] Let (E, Λ) be a fuzzy antinormed linear space with respect to an idempotent t-conorm ◊ satisfying .Then statistical fuzzy λ-antilimit of a fuzzy statistically λ-anticonvergent sequence is unique.

Proof. Let $(x_n)_{n\in N}$ be a fuzzy statistically λ-anticonvergent sequence converging to distinct points x and y in E. This means that for each t > 0 there are two subsets M_x and M_y of N with $\delta(M_x) = 1 = \delta(M_y)$ such that we have $\nu(x_n - x, t/2) < 1 - \lambda$ for each n $\in$ M_x and $\nu(x_n - x,\ t/2)\ ] < 1 - \lambda$ for each n $\in$ M_y. The set $M = M_x \cap M_y$ is sequentially dense in N, and by the assumption on ◊ for each n$\in$ M we have

$$\nu(x-y, t) = \nu(x - x_n + x_n - y, t)$$

$$\leq \nu\left(x_n - x, \frac{t}{2}\right) \Diamond \nu\left(x_n - y, \frac{t}{2}\right) \tag{16}$$

$$< (1-\lambda) \Diamond (1-\lambda) = 1-\lambda$$

Therefore, $\nu(x-y, t) < 1$ for each t > 0. By (FaN6) one obtains $x - y = \theta$, i.e., x = y.

Theorem(4.1.14)[11] Let (E,Λ) be a fuzzy antinormed linear space with respect to an idempotent t-conorm $\Diamond$ satisfying (FaN6). Then:

(i) If $(x_n) \xrightarrow{a-st-\lambda} x$ and $(y_n) \xrightarrow{a-st-\lambda} y$, then $(x_n + y_n) \xrightarrow{a-st-\lambda} x + y$

(ii) If $(x_n) \xrightarrow{a-st-\lambda} x$ and $r \in R$, then $(r \,.\, x_n)$ rx.

Proof. (i) Since $(x_n) \xrightarrow{a-st-\lambda} x$ for each $t > 0$ there is $M_1 \subset N$ with $\delta(M_1) = 1$ such that $\nu(x_n - x, t/2) < 1 - \lambda$ for each $n \in M_1$. Similarly, from$(y_n) \xrightarrow{a-st-\lambda} y$ it follows that for each t>0 there is a set $M_2 \subset N$ with $\delta(M_2)=1$ such that $\nu(y_n - y, t/2) < 1 - M_2$ for each $n \in M_2$. Then $M = M_1 \cap M_2$ is such that $\delta(M)=1$ and for each t>0 and each $n \in M$ we have

$$\nu(x_n + x_n - x - y, t) \leq \nu\left(x_n - x, \frac{t}{2}\right) \Diamond \nu\left(x_n - y, \frac{t}{2}\right)$$

$$< (1 - \lambda) \Diamond (1 - \lambda) = 1 - \lambda, \qquad (17)$$

which means $(x_n + y_n) \xrightarrow{a-st-\lambda} x + y$

(ii) The fact $(x_n) \xrightarrow{a-st-\lambda} x$ implies that for each $t > 0$ there is a statistically dense subset M of N such that $\nu(x_n - y, t) < 1 - \lambda$ for each $n \in M$.Then for each $t > 0$

$$\nu(rx_n - rx, t) = \nu\left(x_n - x, \frac{t}{|r|}\right) < 1 - \lambda \qquad (18)$$

for each $n \in M$; i.e., $(r \,.\, x_n) \xrightarrow{a-st-\lambda} rx$.

Theorem (4.1.15)[11] Let (E, Λ) be a fuzzy antinormed linear space with respect to an idempotent t-conorm $\Diamond$. If $(x_n)_{n \in N}$ is a fuzzy statistically λ-anticonvergent sequence in (E, Λ) statistically converging to $x \in E$, then $||x_n - x||_\lambda$ statistically converges to 0.

Proof. By assumption there is a set $M \subset N$ with $\delta(M) = 1$ such that for each $t > 0$and each $n \in M$ we have $\nu(x_n - x, t) < 1 - \lambda$. In other words, $||x_n - x||_\lambda < t$ for each $n \in M$. Since t was arbitrary we have that $||x_n - x||_\lambda$ statistically converges to 0.

Definition(4.1.16)[11] Let $\lambda \in (0, 1)$. A sequence $(x_n)_{n\in N}$ in a fuzzy antinormed linear space (E, Λ) (with respect to a t-conorm $\lozenge$) is said to be fuzzy statistically λ-anti-Cauchy if for every $t > 0$ there is a set $M \subset N$ such that $\delta(M) = 1$ and for all $m, n \in M$, $\nu(x_n - x_m, t) < 1 - \lambda$.

Theorem(4.1.16)[11] Let (E, Λ) be a fuzzy antinormed linear space with respect to an idempotent t-conorm $\lozenge$ satisfying (FaN6) and1 $\lambda \in (0,1)$.Then every fuzzy statistically λ-anticonvergent sequence $(x_n)_{n\in N}$ in $(E, \Lambda, \lozenge)$ is fuzzy statistically λ-anti- Cauchy.

Proof. Since $(x_n)_{n\in N}$is fuzzy statistically λ-anticonvergent to some $x \in E$, for each $t > 0$, there is a set $M \subset N$with $\delta(M) = 1$ such that $\nu(x_n - x, t/2) < 1 - \lambda$ for each $n \in M$. Then for all $m, n \in M$ we have

$$\begin{aligned} \nu(x_n - y_m, t) &= \nu(x_n - x + x - x_m, t) \\ &\le \nu\left(x_n - x, \frac{t}{2}\right) \lozenge \nu\left(x_m - y, \frac{t}{2}\right) \qquad (19) \\ &< (1 - \lambda) \lozenge (1 - \lambda) = 1 - \lambda. \end{aligned}$$

which means that $(x_n)_{n\in N}$ is fuzzy statistically λ-anti-quasi- Cauchy in (E, Λ).

Theorem(4.1.17)[11] Let (E,Λ) be a fuzzy antinormed linear space with respect to an idempotent t-conorm $\lozenge$.Then every statistically Cauchy sequence$(x_n)_{n\in N}$ in $(E, ||.||_\lambda)$, $\lambda \in (0, 1)$, is fuzzy statistically λ-anti-quasi-Cauchy in (E, Λ).

Proof. Let $\lambda \in (0,1)$ be arbitrary and fixed. Since $(x_n)_{n\in N}$is a statistical Cauchy sequence in $(E, ||.||_\lambda)$, for any $\varepsilon > 0$ there is a setM $\subset$ N with $\delta(M) = 1$ such that for all m, n $\in$ M we have $||x_n - x_m||_\lambda < \varepsilon$. It means that for each m, n $\in$ M

$$\wedge\{t > 0 : \nu(x_n - x_m, t) \leq 1 - \lambda\} < \varepsilon \qquad (20)$$

which implies the existence of $t_0 < \varepsilon$ such that $\nu(x_n - x_m, t_0) < 1-\lambda$. It follows that $\nu(x_n - x_m, \varepsilon) < 1-\lambda$, and as ε was arbitrary, we conclude that $\nu(x_n - x_m, t) = 1 - \lambda$ for each $t > 0$ and all m, n $\in$ M. This means that $(x_n)_{n\in N}$ is fuzzy statistically λ-anti-quasi-Cauchy sequence in (E, Λ). But, λ also was an arbitrary element in (0, 1) so thatwe have that $(x_n)_{n\in N}$ is fuzzy statistically λ-anti-quasi-Cauchy in (E, Λ) for each $\lambda \in (0, 1)$.

Definition(4.1.18)[11] A fuzzy antinormed linear space (E, Λ) (with respect to a t-conorm◊) is said to be fuzzy statistically λ-anticomplete, $\lambda\in (0, 1)$, if every fuzzy statisticallyλ-anti-Cauchy sequence in E fuzzy statistically λ-anticonverges in E.

Theorem(4.1.19)[11] Let (E,Λ) be a fuzzy antinormed linear space with respect to an idempotent t-conorm ◊. If (E, Λ) is fuzzy statistically λ-anticomplete, then E is statistically complete with respect to $||.||_\lambda$ for each $\lambda \in (0, 1)$.

Proof. Let $\lambda \in (0, 1)$ be fixed and let $(x_n)_{n\in N}$ be a statistically Cauchy sequence in E with respect to $||.||_\lambda$. By the previous theorem $(x_n)_{n\in N}$ is fuzzy statistically λ-anti-Cauchy in (E, Λ). Therefore, there is x $\in$ E and a subset M of N with $\delta(M) = 1$ such that, for each $t > 0$ and each n $\in$ M, $\nu(x_n - x, t) < 1-\lambda$. By Theorem (4.1.20), this means $||x_n - x||_\lambda$ statistically converges to 0; i.e., $(x_n)_{n\in N}$ statistically converges to x with respect to $||.||_\lambda$. Therefore, $(E, ||.||_\lambda)$ is statistically complete.

Section (4-2) Some Covering Properties

Let$(E,\nu,\lozenge)$ be a fuzzy antinormed linear space where $\lozenge$ is idempotent.

Given $x\in E$, $\varepsilon \in (0, 1)$, and $t > 0$, the set

$$B_\nu (x, \varepsilon, t) = \{y \in E: \nu(y - x, t) < \varepsilon\} \quad (21)$$

Is called the open ball with center x and radius ε with respect to t.

For each point $y \in B_\nu (x, \varepsilon, t)$there is an open ball with center y contained in $B_\nu (x, \varepsilon, t)$. Let $\nu(y - x, t) = \alpha < \varepsilon$ and set $\beta = \varepsilon - \alpha$. We prove $B_\nu (x, \varepsilon, t/2) \subset B_\nu (x, \varepsilon, t)$. Let $p \in B_\nu (y, \beta, t/2)$. Then $\nu(p - y, t/2) < \beta$ so that we have

$$\begin{aligned}\nu(p-x,t) &= \nu\left(p-y+y-x,\frac{t}{2}+\frac{t}{2}\right)\\ &= \nu\left(p-y,\frac{t}{2}\right) \lozenge\ \nu\left(y-p,\frac{t}{2}\right) \quad (22)\\ &< (\varepsilon-\alpha)\lozenge\varepsilon \le \varepsilon\lozenge\varepsilon = \varepsilon,\end{aligned}$$

i.e., $p \in B_\nu (x, \varepsilon, t)$.

Therefore, the collection

$$\{B (x, \varepsilon, t): x \in E, \varepsilon \in (0, 1) , t > 0\} \quad (23)$$

is a base of a topology on E; denote this topology by τ_ν. Notice that the collection

$$\left\{B\left(x,\frac{1}{n},t\right): x\in E, n\in N, t>0\right\} \quad (24)$$

is also a base for τ_ν. The topology τ_ν is Hausdorff and first countable.

The following definitions are motivated by definitions of the classical Menger, Rothberger, and Hurewicz covering properties.

Definition(4.2.1)[11] A fuzzy antinormed linear space (E, ν, ◊) is said to be

M: Menger-bounded (or M-bounded),

R: Rothberger-bounded (or R-bounded),

H: Hurewicz-bounded (or H-bounded)

if for each sequence (ε_n: n ∈ N) of elements of (0, 1) and each t > 0 there is a sequence

M: (A_n: n ∈ N) of finite subsets of E such that

$E= \bigcup_{n\in N}\bigcup_{a\in A_n} B(a, \varepsilon_n, t)$,

R: (x_n: n ∈ N) of elements of E such that

$E = \bigcup_{n\in N}B(x_n, \varepsilon_n, t)$,

H: (A_n: n ∈ N) of finite subsets of E such that for each x∈ E there is n_0∈ N such that $x \in \bigcup_{a\in A_n} B(a, \varepsilon_n, t)$for all $n \geq n_0$.

A fuzzy antinormed linear space (E, ν, ◊) is said to be precompact (respectively, pre-Lindelof) if for every $\varepsilon \in (0, 1)$ and every t>0 there is a finite (respectively, countable) set A⊂Esuch that E $= \bigcup_{a\in A} B(a, \varepsilon_n, t)$.

Evidently,

$$\text{precompact} \Rightarrow \text{H} - \text{bounded} \Rightarrow \text{M} - \text{bounded} \Rightarrow \text{pre} - \text{Lindelof},$$

$$\text{R} - \text{bounded} \Rightarrow \text{M} - \text{bounded}. \qquad (25)$$

Example(4.2.2)[11] Let (E, ||. ||) be a normed linear space with the Menger (Rothberger,Hurewicz) property. Consider the fuzzy antinormed linear space (E, ν, ◊), where] and ◊ are as in

Example(4.2.3)[11] Then this fuzzy antinormed linear space is M bounded (R-bounded, H-bounded). Consider only the M-bounded case because the other two are shown quite similarly.

Let (ε_n: n ∈ N) be a sequence in (0, 1) and let t> 0. As (E, ||. ||) has theMenger covering property, there is a sequence (A_n: n ∈ N) of finite subsets of X such that

$$E = \bigcup_{n\in N} \bigcup_{a\in A_n} K(a, \varepsilon_n), \tag{26}$$

Where $K(a,\varepsilon_n)$ = {y ∈ X : ||a – y|| < ε}.

Let x ∈ X. There is n ∈ N and a point a_n∈ A_n satisfying $||x – a|| < \varepsilon_n$. Then $\nu(x - a_n, t)$

$$= \begin{cases} 0, & \text{if } t > \|x - a_n\|; \\ \dfrac{\|x - a_n\|}{t\|x - a_n\|}, & \text{if } t \le \|x - a_n\|,\ t > 0; \end{cases} \tag{27}$$

If t > $\|x-a_n\|$, then $\nu(x - a_n, t) = 0$ so that x ∈ B_ν (a_n, ε_n,t)). If t > 0 and t ≤ $\|x-a_n\|$, then $\nu(x - a_n, t) = (\|x-a_n\|/(t + \|x-a_n\|))\varepsilon_n$; i.e., in this case also x ∈ B_ν (a_n, ε_n,t).

Therefore, E = $\bigcup_{n\in N}\bigcup_{a\in A_n}$ B_ν (a, ε_n,t); i.e., (E, ν, ◊) is M bounded.

Example(4.2.4)[11] Let E = R and || . || = | . |, and let ◊ be defined as in then the fuzzy antinormed space (E, ν, ◊) is M-bounded by the previous example and the fact that (R, | . |) has the Menger property. On the other hand, (E, ν, ◊) is not R-bounded.

Indeed, if we take the sequence (2^{-n} : n ∈ N) ⊂ (0, 1) and t = 2^{-1}, then X cannot be covered by the open balls B(x_n, 2^{-n}, 2^{-1}) for any choice of elements x_n, n ∈ N, from E. Otherwise, we would have that for every x ∈ R the following holds:

$$\frac{|x - x_n|}{2^{-1} + |x - x_n|} < 2^{-n}, \tag{28}$$

which means

$$R = \bigcup_{n \in N} \left(x_n - 2^{-n}, x_n + 2^{-n} \right). \qquad (29)$$

However, it is impossible.

We end with the following result on M-boundedness.

If (E, ν, ◊) is a fuzzy antinormed space and S ⊂ E, then (S, ν_s, ◊), where $\nu_s = \nu \restriction (S \times R)$, is also a fuzzy antinormed space and it is called the fuzzy antinormed subspace of (E, ν, ◊).

Section (4-3) Some Topological Properties of Extended b-Metric Space.

In 1993 Czerwik introduced the concept of a b-metric space by giving an axiom wich was weaker than the triangular inequality as follows.

Definition(4.3.1)[14] Let X be a nonempty set and A function d: X × X⟶[0, +∞[is called a b-metric if for all x, y, z e X it satisfies

i. d(x, y) = 0 ⇔ x = y

ii. d (x, y) = d (y, x)

ii. d(x,z) < 2[d(x,y) + d(y, z)]

After that in 1998 Czerwik generalized this notion where the onstant ii was replaced by a constant b ≥1. In 2017, Kamran, Samreen, U_l Ain generalized this notion as following

Definition(4.3.2)[14] Let X be a nonempty set and θ:X×X⟶[1, +∞], A function d_θ : X × X ⟶ [0, +∞] is called an extended b-metric if for all x, y, z ∈ X it satisfies

i. d_θ(x, y) = 0 ⇔ x = y

ii. d_θ (x, y) = d_θ (y, X)

ii. d_θ (x, z) ≤ θ(x, z) [d_θ (x, y) + d_θ(y, z)]

Example(4.3.3)[14] Let X ={2,3,4}, Define θ: X × X ⟶[1, +∞[and d_θ: X×X ⟶[0, +∞[such that

$$\theta(x, y) = 2 + x + y,$$

$$d_\theta(2,2) = d_\theta(3,3) = d_\theta(4,4) = 0$$

$$d_\theta v(2,3) = d_\theta(3,2) = 30 \; , \; d_\theta(2,4) = d_\theta(4,2) = 200 \; ,$$

$$d_\theta\,(3,4) = d_\theta\,(4,3) = 2000$$

Example(4.3.4)[14] Let X=C([a,b],R) be the space of all continuous real valued functions defined on [a, b], X is an extended b-metric space by considering

$$d_\theta(\,x\;y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2 \text{ with}$$

$$\theta(x, y) = |x(t)| + |y(t)| + 1, \text{ where } g: X \times X \longrightarrow [1, +\infty],$$

It is obvious that the class of extended b-metric spaces is larger than b-metric spaces, because if $\theta(x, y) = b$, for $b \geq 1$ then we obtain the definition of a b-metric space.

Definition (4.3.5)[14] Let (X, dθ) be an extended b-metric space,

(i) A sequence $\{x_n\}$in X is said to converge to x∈X, if for every ε>0 there exist N=N(ε)∈N such that $d_\theta(x_n, x)<\varepsilon$, for all $n \geq N$.

(ii) A sequence $\{x_n\}$in X is said to be Cauchy, if for every $\varepsilon > 0$ there exist N=N(ε) ∈N such that $d_\theta(x_N, x_m) < \varepsilon$, for all n,m ≥ N.

(III) An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.

Lemma(4.3.6)[14] Let (X, d_θ) be an extended b-metric space, If d_θ is continuous then every convergent sequence has a unique limit.

main Results(4.3.7)[14] Denote B(a,r) = {y∈X :d_θ(a,y) < r}

B[a,r]=(y∈X :d_θ(a,y) < r} and call them respectively the open ball and the closed ball.

Definition(4.3.8)[14] Let (X, d_θ) be an extended b-metric space ,

i. A subset A of X is called open if for any $a \in A$, it exists $\varepsilon > 0$, such that $B(a, r) \subset A$.

ii. A subset B of X is called close if for any sequence x_N ,such that $\lim_{n\to\infty} x_n = x$ and $x_n \in B$ for all n, then $x \in B$, It is easy to prove the following Lemma.

Lemma (4.3.9)[14] A subset A of X is open if and only if $A^c = X - A$ is closed, In a b-metric space (X,d) are well known the following results

i. d is not necessarily continous in each variable

ii. An open ball is not necessarily an open set.

In an extended b- metric space (X, d_θ) we can say the same thing, since every b- metric space is an extended b-metric space.

Proposition (4.3.10)[14] Let (X, d_θ) be an extended b-metric space, If d_θ is continous in one variable then d& is continous in the other variable.

Proof:

Without loss of generality ,we may assume that d_θ is continuous with respect to the first variable. For each $x \in X$, if $\lim_{n\to\infty} y_n = y$ then we have that,

$$\lim_{n\to\infty} d_\theta(x, y_n) = \lim_{n\to\infty} d_\theta(y_n, x) = d(y, x) = d(x, y).$$

Proposition (4.3.11)[14]

Let (X, d_θ) be an extended b-metric space, If d_θ is continous in one variable then for each $a \in X$ and $r > 0$ we have

i. B(a,r) is open

ii. B[a,r] is closed

Proof: For the first one by using Lemma (4.2.10) we will show that the set $(B(a, r))^c$ is a closed set. Let $\{x_n\}_{n\in N} \subset (B(A,R))^C$ be a sequence such that $\lim_{n\to\infty} x_n = x$. For all $n \in N$ we have $d_\theta\{x_n, a) \geq r$ then $d_\theta(x,a) = \lim_{n\to\infty}(x_n, a) \geq r$, therefore $x \in (B(a, r))^c$, so (B(a, r))c is a closed set.

For the second one, let $\{x_n\}_{n\in N} \subset B[a, r]$ and let $\lim_{n\to\infty} n = x$. For all $n\in N$ we have $d_\theta\{x_n, a) \leq r$.

Then $d_\theta(x, a) = \lim_{n\to\infty} d_\theta(x_n, a) \leq r$. It implies that $x \in B[a, r]$. Therefore B[a, r] is closed.

Propostion (4.3.12)[14] Let $(X_1, d_{\theta_1}),(X_2, d_{\theta_2}),(X_3, d_{\theta_3}),\ldots,(X_n , d_{\theta_n})$ be n extended b-metric spaces. Put

$$d_\theta : X_1 \times X_2 \times \ldots \times X_n \longrightarrow [0, +\infty[\text{ such that}$$

$$d_\theta((x_1, x_2,\ldots x_n),(y_1, y_2,\ldots y_n)) = \sum_{i=1}^{n} d\theta_i(x_i, y_i) \text{and}$$

$$\theta: X_1 \times X_2 \times \ldots \times X_n \longrightarrow [0, +\infty[\text{ such that}$$

$$\theta((x_1, x_2,\ldots x_n),(y_1, y_2,\ldots y_n)) = \max(\theta_i(x_i, y_i)\}, \text{ Then}$$

d_θ is an extended h metric on $X_1 \times X_2 \times \ldots \times X_n$.

Proof: We only give the proof for the product of two extended b-metric space. So let $(X_1, d_{\theta_1}),(X_2, d_{\theta_2})$ be two extended b-metric space. Put

$$d_\theta ((x_1, x_2), (y_1, y_2)) = d_{\theta_1}(x_1, y_1) + d_{\theta_2}(x_2, y_2) \text{ and}$$

$$\theta((x_1, x_2),(y_1, y_2))=\max \{\theta_1(x_1, y_1), \theta_2(x_2, y_2)\}$$

i. $d_\theta((x_1, x_2), (y_1, y_2)) = 0 = d_{\theta_1}(x_1, y_1) + d_{\theta_2}(x_2, y_2)$ if and only if $d_{\theta_1}(x_1, y_1) + d_{\theta_2}(x_2, y_2)= 0$, but since $d_{\theta_1}, d_{\theta_2}$ are extended b-metric space then $x_1 = y_1$, $x_2 = y_2$, hence $(x_1, x_2) + (y_1, y_2)$.

ii. $d_\theta((x_1, x_2), (y_1, y_2)) = d_{\theta_1}(x_1, y_1) + d_{\theta_2}(x_2, y_2) = d_{\theta_1}(x_1, y_1) + d_{\theta_2}(x_2, y_2) = d_\theta((x_1, x_2), (y_1, y_2))$.

iii. Now let us show the following inequality

$\theta((x_1,x_2), (a_1, a_2)) <$

$\theta((x_1, x_2), (y_2, y_2))[d_\theta((x_1, x_2), (z_1, z_2))+d_\theta((z_1, z_2), (y_1, y_2))]$ Since

$d_\theta((x_1, x_2), (y_2, y_2)) = d_{\theta_1}(x_1, y_1) + d_{\theta_2}(x_2, y_2) \leq$

$\theta_1(x_1,y_1)[d_{\theta_1}(x_1, z_1)+d_{\theta_1}(z_1, y_1)]+\theta_1(x_2, y_2)[d_{\theta_2}(x_2, z_2)+d_{\theta_2}(z_2, y_2)]\leq$

$\max\{\theta(x_1,x_2),(y_1,y_2)\}\{d_{\theta_1}(x_1,z_1)+d_{\theta_1}(z_1,y_1)+d_{\theta_2}(x_2,z_2)+d_{\theta_2}(z_2,y_2)\}=$

$\theta((x_1,x_2),(y_1,y_2))[d_\theta((x_1, x_2), (z_1, z_2))+d_\theta((z_1, z_2), (y_2, y_2))]$

Proposition (4.3.13)[14]

The topology induced by d& and the product topology of $X_1 \times X_2$ are coincident.

Proof: We will show that for all $(a_1, a_2) \in X_1 \times X_2$ the following inclusions holds

i. $B((a_1, a_2), r) \subseteq B(a_1, 2r) \times B(a_2, 2r)$

ii. $B(a_1, r) \times B(a, r) \subseteq B((a_1, a_2), 2r)$

Let $(x_1, x_2) \in B((a_1, a_2), r)$ then

$d_\theta((x_1, x_2), (a_1, a_2)) = d_{\theta_1}(x_1, a_1) + d_{\theta_2}(x_2, a_2) < r$. So we have that $d_{\theta_1}(x_1, a_1) \leq r < 2r$ and $d_{\theta_2}(x_2, a_2) \leq r < 2r$, therefore $x_1 \in B(a_1, 2r)$ and $x_2 \in B(a_2, 2r)$ hence $(x_1, x_2) \in B(a_1, 2r) \times B(a_2, 2r)$

We can prove similarly the second inclusion.

Corollary (4.3.14)[14] $\lim_{n\to\infty}(x_n^1, x_n^2) = (x_1, x_2)$ in $(X_1, \times X_2, d_\theta)$ if and only if $\lim_{n\to\infty} x_n^1 = x_1$ in(X_1, d_{θ_1}) $\lim_{n\to\infty} x_n^2 = x_2$ in (X_1, d_{θ_2}).

Corollary (4.3.15)[14] A sequence $\{(x_n^1, x_n^2)\}$ is Cauchy in $(X_1, \times X_2, d_\theta)$ if and only if $\{(x_n^1)\}$ is Cauchy in (X_1, d_θ) and $\{(x_n^2)\}$ is Cauchy in (X_2, d_θ).

Corollary (4.3.16)[14] $(X_1, \times X_2, d_\theta)$ is complete if and only if (X_1, d_{θ_1}) and (X_2, d_{θ_2}) are complete.

Definition (4.3.17)[14] A subset U of X is sequentially open if each sequence (x_n) in X that converges to a point x of U, then it exist N ∈ N such that $x_n \in$ N for each n > N.

Definition(4.3.18)[14] Let $A \subset B_\varepsilon$ be an extended b-metric space. And C a subset of X

i. C is compact if and only iffor every sequence of elements of C there exists a subsequence that converges to an element of C.

ii.C is bounded if and only if $\delta(C) = \sup\{d_\theta(x,y) : x,y \in C\} < \infty$

Let H(X) denote the set of all nonempty compacts subsets of X. For A, B ∈ H(X), let

$$H(A, B) = \max\left\{\sup_{a\in A}\left(d_\theta(a,B)\right), \sup_{a\in B}\left(d_\theta(b,A)\right)\right\}$$

Where $d_\theta(x, B) = \inf\{d_\theta(x, y): y \in B\}$ is the distance of a point x from the set B. The mapping H is said to be the Pompeiu-Hausdorff metric induced by d_θ.

For any A∈H(X), and any positive number ε, let

$A_\varepsilon = \{x \in X: d_\theta(x, y) \le \varepsilon, \text{for some } y \in A\} = \{x \in e\ X: d_\theta(x, A) \le \varepsilon\}$.

Remark (4.3.19)[14]

i .Notice that the infimum in the definition of d_θ (x, B) is actually achieved, that is there is some point y of B such that $d_\theta(x, B) = d_\theta(x, y)$, since B is compact.

ii. $\sup_{a\in A}(d_\theta(a, B)) \le \varepsilon$ if and only if $A \subset B_\varepsilon$. By this last one we can give an equivalent definition for the mapping H as following $H(A, B) = \inf\{\varepsilon: A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}$

Proposition(4.3.20)[14] Let (X, d_θ) be an extended b-metric space, For any A,B,C,D sets of H(X) we have

ii. $\sup_{a\in A}(d_\theta(a, B)) = 0$ if and only if $A \subseteq B$

ii.If $B \subseteq C$ then $\sup_{a\in A}(d_\theta(a, C)) \le \sup_{a\in A}(d_\theta(a, B))$

ii. $H(A\cup B,C\cup D) \le \max\{H(A,C),H(B,D)\}$

Proof: It is easy to prove the first property ε

i. It is clear that since $B \subseteq C$ then $B_\varepsilon \subseteq C_\varepsilon$. Let suppose that $\sup_{a\in A}(d_\theta(a, B)) = \varepsilon$**.** By Remark(4.3.19) we have that $A \subset B_\varepsilon \subseteq C_\varepsilon$, that is $\sup_{a\in A}(d_\theta(a, C)) \le \varepsilon = \sup_{a\in A}(d_\theta(a, B))$

ii.Without losing generality we suppose that

$\max\{H(A, C)\ H(B, D)\} = H(A, C) = \varepsilon$. Then $A \subset C_\varepsilon$ and $C \subset A_\varepsilon$, also we have that H(B,D) H(A, C),hence $B \subset D_\varepsilon$ and $D \subset B_\varepsilon$. This implies that $A\cup B \subset A_\varepsilon \cup D_\varepsilon = (C \cup D)_\varepsilon$ and $C \cup D \subset A_\varepsilon \cup B_\varepsilon = (A \cup B)_\varepsilon$, which means that

$$H(AB,C\cup D) \le \varepsilon = H(A,C) = \max\{H(A, C), H(B,D)\} .$$

Theorem (4.3.21)[14] Let (X, d_θ) be an extended b-metric space and CB(X) denote the set of all closed and bounded subsets of X, Then (CB(X), H_θ) is an extended b- metric space where the mapping

$$\theta : CB(X) \times CB(X) — [l, +\infty) \text{ is such that}$$
$$\theta(A, B) = \sup \{\theta(a, b): a \in A, b \in B\}$$

Proof: To prove that H_θ is an extended b-metric we need to verify the following four properties

i. $H(A, B) \geq 0$ for all $A, B \in CB(X)$

ii. $H(A,B) = 0$ if and only if $A = B$

ii. $H(A, B) = H(B, A)$ for all $A, B \in CB(X)$

iv. $H(A, B) \leq \theta(A, B)[H(A, C) + H(C, B)]$

The first property is trivial since $\sup_{a\in A}(d_\theta (a, B))$ and $\sup_{a\in A}(d_\theta (b, A))$ are nonnegative.For the second property, suppose A=B. Therefore from property 1 of Proposition (4.3.20), $A \subseteq B$ and $B \subseteq A$, hence $\sup(dg (a, B)) =0$ and $\sup_{a\in A}(d_\theta(b, A)) =0$. Thus $H (A, B) = 0$. Now suppose $H (A, B) = 0$. This implies $\sup_{a\in A}(d_\theta(a, B)) =0$ and $\sup_{a\in A}(d_\theta(b, A)) =0$. Again by property 1 of Propositon (4.3.20) it follows that $A\subseteq B$ and $B \subseteq A$, hence $A = B$. The third property can be proved by the symmetry of the definition of H(A,B). For the final property by using the property 1 of Remark (4.3.19) we have that for each $a\in A$ there exist an element $c_a\in C$ such that $d_\theta(a, C) = d_\theta (a, c_a)$. We then have that

$$d_\theta(a,B) = \inf \{d_\theta(a,b): b \in B\} \leq$$
$$\inf \{\theta(a, b)[d(a, c_a) + d(c_a,b)]: b \in B\} =$$
$$\theta(a,b)d(a, c_a) + \inf \{\theta(a,b)d(c_a,b): b \in B\} \leq$$
$$\theta(A, B)d(a, c_a) + \inf \{\theta(A, B)d(c_a,b) : b \in B\} =$$
$$\theta(A,B)d(a,c_a) + \theta(A,B) \inf \{d(c_a,b): b \in B\} =$$
$$\theta(A, B)d_\theta(a, C) + \theta(A, B)d_\theta(c_a, B) \leq$$
$$\theta (A, B)[\sup_{a\in A}(d_\theta(a, C)) + \sup_{c\in C}(d_\theta(c, B))]$$

Since $a\in A$ was arbitrary, by taking the supremum of $d_\theta(a, B)$ we get that $\sup_{a\in A}((a, B)\leq \theta(A, B)[\sup_{c\in C}(d_\theta (a, C)) + \sup(dg (c, B))] \leq$

$$G(A, B)[H (B, C) + H (C, A)]$$

Analogusly we can show that

$$\sup_{b\in B}(d_\theta(b, A)) \leq g(A, B)[\sup_{c\in C}(d_\theta(b, C)) + \sup(dg (c, A))]\leq$$
$$\theta(A, B)[H (B, C) + H (C, A)]$$

Hence we get that

$$H (A, B) \leq \theta(A, B)[H (A, C)+H (C, B)]$$

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