UNIVERSITY OF NILE VALLEY

COLLEGE OF GRADUATE STUDIES

ON THE HYDRODYNAMIC PRPBLEMS

A THESIS SUBMITTED FOR PARTIALY
FULFILLMENT OF REQUIREMENTS

FOR M.SC.IN MATHEMATICS

BY :

OSMAN ABDALLAH ADAM OSMAN

SUPERVISED BY :

Dr. ADAM ABDALLAH ABBAKER HASSAN

August 2009

Dedication

To my father , mother ,wife and my sons .

For patience , encouragement and support .

To my brother : Arabe Ali Aamer and his family members.

Acknowledgment

I am deeply indebted to my supervisor Dr / ADAM ABDALLAH ABBAKER who suggested the title of this thesis , for his continuous support , encouragement valuable suggestions , constructive criticism , patience and wise guidance .

Special thanks are also due to the University of Khartoum Library staff for their patience and continuous help .

Abstract

We shall consider in this Research the application of the theory of integral transforms to the solution of boundary value problems in hydrodynamics. The field of hydrodynamics is so vast that we can consider only a few representative problems which illustrate the methods of solution involved. Before discussing the solution of special problems we shall briefly outline the general theory underlying the establishment of the equation of motion of a fluid.

The sub category fluid mechanics is defined as the science that deals with the behavior of fluids at rest (fluid statics) or in motion (fluid dynamics) and interaction of fluid .

We shall consider in chapter one some history of the hydrodynamic and We start the equation of continuity and equation of motion and the vorticity of equation .

In chapter two We start Irrotational flow of a perfect fluid and the Irrotationaly potential flow and two – dimensional flow and the steady flow of perfect fluid through a slit , and it also flow of perfect fluid through a circular aperture in a plane rigid screen .

In chapter three We shall consider the surface wave and the surface waves generated by an impulsive pressure and the wave – propagation in two dimensionas and that slow motion of a viscous fluid and also Diffusion of vorticity .

In chapter four We consider the motion of a viscous fluid contained between two Infinite coaxal cylinders and the motion when the outer cylinder rotates at a constant speed , and the motion of a viscous fluid under a surface load , and We shall briefly consider the Harmonic Analysis of Nonlinear viscous flow ,

and in the last chapter We stad the stability of theory of hydrodynamic .

الخلاصة

في هذا البحث تم استعراض نظرية التحويل التكاملية في حل مشاكل قيمة الحد في الهيدروديناميكا .

أن حقل الهيدروديناميكا واسع جداً بحيث أن نعتبر فقط بضعة مشاكل تمثيلية التي تصور طرق الحل المعقد .

قبل مناقش حل المشاكل الخاصة التي نحن سنلخص النظرية العامة سريعاً تقع تحت مؤسسة معادلة حركة السوائل .

ان ميكانيكا صنف الغواصة السائلة معرفه كالعلم الذي يتعامل مع سلوك السائل في الاستراحة (علم توازن قوى السائل) أو في الحركة (دنياميكا سائلة) وتفاعل السائلة.

قدم في الباب الأول بعض من تاريخ الهيدروديناميكي مثل معادلة الاستمرارية ومعادلة الحركة ومعادلة السرعة .

في الباب الثاني تمت مناقشة تدفق متقن للسائل وتدفق محتمل وتدفق في البعد الثنائي وأيضا التدفق الثابت للسائل المثالي خلال الشق وكذلك تدفق السائل المثالي خلال فتحة دائرية في شاشة صارمة مستوية .

كما أن الباب الثالث استعرض الموجة السطحية والموجات السطحية ولدا بضغط مندفع وتوليد الموجة في البعد الثنائي وذلك ببطئ حركة السائل اللزج وأيضا انتشارها

كما ناقش الباب الرابع حركة السائل اللزج بين اسطوانتين محوريتين لانهائيتين . والحركة عندما تدور الاسطوانة الخارجية في سرعة ثابتة وحركة السائل اللزج تحت حمل سطحي وايضاً التحليل التوافقي من تدفق لزج لا خطي سريعا. وفي النهاية تم دراسة استقرار نظرية الهيدروديناميكي .

CONTENTS

Chapter Four Fluid Motion

Introduction

Mechanics is the oldest physical science that deals with both stationary and moving bodies under the influence of forces. The branch of mechanics that deals with bodies at rest is called statics, while the branch that deals with bodies in motion is called dynamics. [4]

The subcategory fluid mechanics is defined as the science that deals with the behavior of fluids at rest (fluid statics) or in motion (fluid dynamics) and the interaction of fluids with solids or other fluids at the boundaries. [4]

Fluid mechanics is also referred to as fluid dynamics by considering fluids at rest as a special case of motion with Zero velocity. [4]

Fluid mechanics itself is also divided into several categories. The study of the motion of fluids that are practically incompressible (such as liquids, especially water, and gases at low speeds) is usually referred to as hydrodynamics. [4]

A sub category of hydrodynamics is hydraulics which deals with liquid flows in pipes and open channels. Gas dynamics deals with the flow of fluids that undergo significant density changes, such as the flow of gases through nozzles at high speeds. The category aerodynamics deals with the flow of gases (especially air) over bodies such as aircraft, rockets, and automobiles at high or low speeds. Some other specialized categories such as meteorology, oceanography and hydrology deal with naturally occurring flows. [4]

Chapter One

Chapter One

Hydrodynamic Equations

(1-1)Some history

The mathematical history of fluid mechanics begins with Leonhard Euler who was invited by Frederick the Great to Potsdam in 1741. [7]

According to a popular story (which we have not been able to corroborate) one of his tasks was to engineer a water fountain. As a true theorist, he began by trying to understand the laws of motion of fluids. in 1755 he wrote Newton's Lows for a fluid which in modern notation reads (for the case of constant density. [7]

$$\frac{\partial \mathrm{u}(\mathrm{r},\mathrm{t})}{\partial \mathrm{t}} + u(r,t)\,.\nabla u(r,t) = -\nabla v(r,t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1.1)$$

Here u(r,t) and v(r,t) are the fluid velocity and pressure at the spatial point (r) at time (t).

The (L H S) of this " Euler equation " for u(r,t) is just the material time derivative of the momentum, and the (R H S) is the force, which is represented as the gradient of the pressure, imposed on the fluid. In fact , trying to build a fountain on the basis of this equation was bound to fail. [7]

This equation predicts, for a given gradient of pressure , velocities that are much higher than anything observed. One missing idea was that of the viscous dissipation that is due to the friction of one parcel of fluid against neighboring ones. The appropriate term was added to (1) by Navier in 1827 and by Stokes in 1845-2- [7]

The result is known as the " Navies – Stokes equation " :

$$\frac{\partial \mathrm{u(r,t)}}{\partial \mathrm{t}} + u(r,t)\nabla u(r,t)$$
$$= -\nabla v(r,t) + v\nabla^{2} u(r,t) \ldots\ldots\ldots (1.1.2)$$

Here (v) is the kinematic viscosity , which is about 10^{-2} and o.15 cm^2/sec for water and air at room temperature respectively. Without the term $v\nabla^2 u(r,t)$ the kinetic energy $u^2/2$ is conserved; with this term kinetic energy is dissipated and turned into heat. The effect of this term is to stabilize and control the nonlinear energy conserving Euler equation. [1]

Straightforward attempts to assess the solutions of this equation may still be very non-realistic. For example, we could estimate the velocity of water flow in any one of the mighty rivers like the Nile or the Volga which drop hundreds of meters in a course of about a thousand kilometers.

The typical angle of inclination is about 10^{-4} radians, and the typical river depth (L) is about 10 meters. Equating the gravity force $\propto$ g (g$\simeq \frac{10^3 Cm}{sec^2}$) and the viscous drag ($vd^2u/dz^2 \sim v\,u/L^2$) we find (u) to be of The order of 10^7 cm / sec instead of the observed value of about 10^2cm/sec. This is of course absurd; perhaps to the regret of the white water rafting industry. This estimate contradicts even simple energy conservation arguments. After all, we cannot gain in kinetic energy more than the stored potential energy which is of the order of (pgH) where (H) is the drop in elevation of the river bed from its source. For the Volga or the Nile (H) is about $5X10^4$ cm, and equating the potential energy drop with the kinetic energy we estimate (u $\sim\sqrt{2gh} \simeq 10^4$cm/sec).

This is still off the mark by two orders of magnitude.

The resolution of this discrepancy was suggested by Reynolds

who stressed the importance of a dimensionless ratio of the nonlinear term to the viscous term in (2). With a velocity drop off the order of (U) on a scale (L) the non liner term is estimated as u^2/L. The viscous term is about $(v\,u/L^2)$. the ratio of the two, known as the Reyndds number Re, is (UL/ v). the magnitude of Re measures how large is the nonlinearity compared to the effect of the viscous dissipation in a particular fluid flow.

For Re << 1 one can neglect the nonlinearity and the solutions of the Navier-Stokes equations can be found in closed-form in many instances. In many natural circumstances Re is very large. For example, in the rivers discussed above Re $\simeq 10^7$. Reynolds understood that for Re >> 1 there is no stable stationary solution for the equations of motion.

The solutions are strongly affected by the nonlinearity, and the actual flow pattern is complicated, convoluted and vertical. Such flows are called turbulent.

Modern concepts about high Re number turbulence started to evolve with Richardson's insightful contributions which contained the famous " Poem" that Paraphrased (J). Swift : " Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity-in the molecular sense " . In this way Richardson conveyed an image of the creation of turbulence by large scale by the nonlearities of fluid motion, until the energy dissipates at small scales by viscosity, turning into heat. This picture led in time to innumerable " cascade models " that tried to capture the statistical physics of turbulence by assuming something or other about the cascade process. Indeed, no one in their right mind is interested in the full solution of the turbulent velocity field at all points in space-time .

The interest is in the statistical properties of the turbulent flow. Moreover the statistics of the velocity field itself is too heavily dependent on the particular boundary conditions of the flow. Richardson understood that universal properties may be found in the statistics of velocity differences.

$\delta u(r1,r2) \equiv u(r_2) - u(r1)$ across a separation

R = $r_2 - r_1$. in taking such a difference we subtract the non-universal large scale motions (known as the " wind" in atmospheric flows). In experiments (see for example (5,6,7,8,9,10) it is common to consider one dimensional cuts of the velocity field, δuL(R) $\equiv$ δu(r_1,r_2). $^R/_R$. The interest is in the probability distribution function of δul(R) and its moments. These moments are known as the " structure function".

$$\text{Sn(R)} \equiv (\delta uL(R)^{\mathrm{n}}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots... (1.1.3)$$

Where < … > stands for a suitably defined ensemble average. For Gaussian statistics the whole distribution function is determined by the second moment S_2(R) , and there is no information to be gained from higher order moments. In contrast, hydrodynamic experiments indicate that turbulent statistic are extremely non-Gaussian, and the higher order moments contain important new information about the distribution functions.

Possibly the most ingenious attempt to understand the statistics of turbulence is due to kolmogorov who in 1941 proposed the idea of universality (turning the study of small scale turbulence from mechanics to fundamental physics) based on the notion of the " interracial range " . the idea is that for very large values of Re there is a wide separation between the " scale of energy input " L and the typical " viscous dissipation scale " at which viscous friction become important and dumps the energy into heat.

In the stationary situation, when the statistical characteristics of the turbulent flow are time independent, the rate of energy in put at large scales (L) is balanced by the rate of energy dissipation at small scales (u), and must be also the same as the flux of energy from larger to smaller scales (denoted $\bar{e}$) as it is measured at any scale (R) in the so-called " inertial " interval (η << R << L).

Kolmogorov proposed that the only relevant parameter in the inertial interval is e^- , and that (L) and (η) are irrelevant for the statistical characteristics of motions on the scale of (R).

This assumption means that (R) is the only available length for the development of dimensional analysis. In addition we have the dimensional parameters ($\bar{e}$) and the mass density of the fluid(ρ). From these three parameters we can form combinations ($P^{\varkappa}$ u^{y} R^{z}) such that with a proper choice of the exponents ($\varkappa, y, z$) we form any dimensionality that we want. This leads to detailed prediction about the statistical physics of turbulence. For example, to predict S_n (R) we not that the only combination of (e^-) and (R) that gives the right dimension for S_n is $(\bar{e}R)^{n/\ni}$. In particular for n = 2 this is the famous kolmogaroy " $2/\ni$" law which in Fourier representation is also known as the " $-5/\ni$ " law. The idea that one extracts universal properties by focusing on statistical quantities can be applied also to the correlations of gradients of the velocity field. An important example is the rate Ɛ(r,t) at which energy is dissipated into heat due to viscous damping. This rate is roughly $\mathrm{v}|\nabla u(r,t)|^2$. One is interested in the fluctuations of the emerge dissipation ∑(r,t) about their mean e¯, e¯(r,t) = Ɛ(r,t)-e¯ , - $\bar{e}$ and how these fluctuations, are correlated in space. The answer is given by the often-studied correlation function

$$Ku(R) =< \ \bar{e}(r+R,t)\ \bar{e}(r,t) > \cdots \dots \dots \dots \dots . . (1.1.4)$$

If the fluctuations at different points were uncorrelated, this function would vanish for all R≠o. Using kolmogorov's dimensional reasoning one estimates $ku(R) \simeq \mathrm{v}2e - 4/\ni R - 8/\ni$, which means that the correlation decays as a power, like $1/R^{8/\ni}$

Experimental measurements show that kolmogorov was remarkably close to the truth. The major aspect of his predictions, i.e. that the statistical quantities depend an the length scale (R) as power laws is corroborated by experiments. On the other hand the predicted exponents seem not to be exactly realized. For example, the experimental correlation K_U(R) decays according to a power law.

$$Ku(R) \sim R^{-\mu} for\ \mu\ <<\ R << L. \dots \dots \dots \dots \dots . (1.1.5)$$

With μ having a numerical value of o.2 − o.3 which is in large discrepancy compared to the expected value of $8/\ni$. The structure function also behave as power laws,

$$Sn(R) \simeq\ R\zeta n\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . . (1.1.6)$$

but the numerical values of Cn deviate progressively from $n/\ni$ when (n) increases . Something fundamental seems to be missing.The uninitiated reader might think that the numerical value of this exponent or another is not a fundamental issue. However one needs to understand that the kolmogorov theory exhausts the dimensions of the statistical quantities under the assumption that $\bar{e}$ is the only relevant parameter. Therefore a deviation in the numerical value of and exponent from the prediction of dimensional analysis requires the appearance of another dimensional parameter. Of course there exist two dimensional parameters , i.e.l and η , which may turn out to be

relevant. Experiments indicate that for the statistical quantities mentioned above the energy-input scale (L) is indeed relevant and it appears as a normalization scale for the deviations from Kolmogorov's predictions :

$$Sn(R\ ) \ \simeq \ (\acute{e}R)^{n/\ni}(L/R)\delta n \ where \ C_n = \ n/\ni -\delta n$$

Such forms of scaling, which deviate from the predictions of dimensional analysis, are referred to as " anomalous scaling " . the realization that the experimental results for the structure functions were consistent with (L) rather than (η) as the normalization scale developed over along time and involved a large number of experiments ; recently the accuracy of determination of the exponents has increased appreciably as a result of aclver method of data analysis by Benzi, Ciliberto and Coworkers. Similarly a careful demonstration of the appearance of (L) in the dissipation correlation was achieved by Sreenivasan and Coworkers. A direct analysis of scaling exponents C_n and (μ) in a high Reynolds number flow was presented by Praskovskii and Oncley, leading to the same conclusions.

Theoretical studies of the universal small scale structure of turbulence can be classified broadly into two main classes . Firstly there is a large collection of phenomenological modes that by

attempting to achieve agreement with experiments have given important insights into the nature of the cascade or the statistics of the turbulent fields . In particular there appeared influential ideas , following Mandelbrot , about the fractal geometry of highly turbulent fields which allow scaling properties that are sufficiently complicated to include non-Kolmogorov scaling . Parisi and Frisch showed that by introducing multifractals one can accommodate the nonlinear dependence of c_n on .
However these models are not derived from the equations of fluid mechanics ; one is always left with uncertainties about the validity or relevance of these models . The second class of approaches is based on the equations of fluid mechanics . Typically one acknowledges the fact that fluid mechanics is a (classical) fields theory and resorts to field theoretic methods in order to compute statistical quantities . Even though there has been a continuous effort for almost 50 years in this direction , the analytic derivation of the scaling laws for $K_u(R)$ and $S_u(R)$ from the Navier-Stokes equations and the calculation of the numerical value of the scaling exponents μ and C_u have been among the most elusive goals of theoretical research . Why did it turn out to be so difficult ?

To understand the difficulties , we need to elaborate a little on the nature of the field theoretic approach . Suppose that we want to calculate the average response of a turbulent fluid at some point r_0 to forcing at point r_1 . The field theoretic approach allows us to consider this response as an infinite sum of all the following processes : firstly there is the direct response at point r_0 due to the forcing at r_1 . This response is caused by linear processes in the fluid , and is instantaneous if we assume that the fluid is incompressible (and therefore the speed of sound is infinite) . Then there are processes which are inherently nonlinear . Nonlinear procesesses are mediated by intermediate points , but take time . Forcing at r_1 causes a response at an intermediate point r_2 , which then acts as a forcing for the response at r_0 . Since this intermediate process can take time , we need to integrate over all the possible positions of point r_2 and all times . This is the second – order term in perturbation theory . Then we can force at r_1 , the response at r_2 acting as a forcing for r_3 and the response at r_3 forces a response at r_0 . We need to integrate over all possible intermediate positions r_2 and r_3 and all the intermediate times . This is the third – order term in perturbation theory . And so on . The actual response is the infinite sum of all these contributions . In applying this field theoretical method one encounters three main difficulties :

(A)The theory has no small parameter . The usual procedure is to develop the theory perturbatively around the linear part of the equation of motion . In other words , the zeroth order solution of Equation (1.1.2) is obtained by discarding the terms which are quadratic in the velocity field . The expansion parameter is then obtained from the ratio of the quadratic to the linear terms ; this ratio is of the order of the Reynolds number Re which was defined above . Since we are interested in $\gg 1$, naive perturbation expansions are badly divergent . In other words the contribution of the various processes described above increases as $(Re)^n$ with the number n of intermediate points in space-time .

(B) The theory exhibits two types of nonlinear interactions . Both are hidden in the nonlinear term $u.\nabla u$ in Equation (1.1.2) . The larger of the two is known to any person who has watched how a small floating object is entrained in the eddies of a river and swept along a complicated path with the turbulent flow . In a similar way any fluctuation of small scale is swept along by all the larger eddies . Physically this sweeping couples any given scale of motion to all the larger scales . Unfortunately the largest scales contain most of the energy of the flow ; these large scale motions are that is experienced as gusts of wind in the atmosphere or the swell in the ocean . In the perturbation theory for $S_n(R)$ One has the consequences of the sweeping effect from all the scales larger than , with the main contribution coming from the largest , most intensive gusts on the scale of L . As a result these contributions diverge when $\to \infty$. In the theoretical jargon this is known as " infrared divergences " . Such divergences are common in other field theories , with the best known example being quantum electrodynamics . In that theory the divergences are of similar strength in higher order terms in the series , and they can be removed by introducing finite constants to the theory , like the charge and the mass of the electron . In the hydrodynamic theory the divergences become stronger with order of the contribution , and to eliminate them in this manner one needs an infinite number of constants . In the jargon such a theory is called " not renormalizable ". However , sweeping is just a kinematic effect that does not lead to energy redistribution between scales , and one may hope that if the effect of sweeping is taken care of in a consistent fashion a renormalizable theory might emerge . This redistribution of energy results from the second type of interaction , that stems from the shear and torsion effects that are sizable only if they couple fluid motions of comparable scales . The second type of nonlinearity is smaller in size but crucial in consequence , and it may certainly lead to a scale–invariant theory .

(C) Nonlocality of interaction in r space . One recognizesthat the gradient of the pressure is dimensionally the same as $(u.\nabla)u$, and the fluctuations in the pressure are quadratic in the fluctuations of the velocity . However , the pressure at any given point is determined by the velocity field everywhere . Theoretically one sees this effect by taking the divergence of Equation (1.1.2) . This leads to the equation $\nabla^2 v = \nabla.[(u.\nabla)u]$. The inversion of the Laplacian operator involves an integral over all space . Physically this stems from the fact that in the incompressible limit of the Navier–Stokes equations sound speed is infinite and velocity fluctuations at all distant point are instantaneously coupled .

Indeed , these difficulties seemed to complicate the application of field theoretic methods to such a degree that a wide – spread feeling appeared to the effect that it is impossible to gain valuable insight into the universal properties of turbulence along these lines , even though they proved so fruitful in other field theories . The present authors (as well as other researchers starting with Kraichnan and recently , Migdal , Polyakov , Eyink etc .) think differently , and in the rest of this paper we will explain why .

The first task of a successful theory of turbulence is to overcome the existence of the interwoven nonlinear effects that were explained in difficulty (B) . This is not achieved by directly applying a formal field – theoretical tool to the Navier – Stokes equations . It does not matter whether one uses standard field theoretic perturbation theory , path integral formulation , renormalization group , expansion , large N-limit or one's formal method of choice . One needs to take care of the particular nature of hydrodynamic turbulence as embodied in difficulty (B) first , and then proceed using formal tools .

The removal of the effects of sweeping is based on Richardson's remark that universality in turbulence is expected for the statistics of velocity differences across a length scale R rather than for the statistics of the velocity field itself . The velocity fields are dominated by the large scale motions that are not universal since they are produced directly by the agent that forces the flow . This forcing agent differs in different flow realizations (atmosphere , wind tunnels , channel flow etc .) . Richardson's insight was developed by Kraichnan who attempted to cast the field theoretic approach in terms of Lagrangian paths , meaning a description of the fluid flow which follows the path of every individual fluid particle . Such a description automatically removes the large scale contributions . Kraichnan's approach was fundamentally correct , and gave rise to important and influential insights in the description of turbulence , but did not provide transparent rules on how to consider all the orders of perturbation theory . The theory did not provide transparent rules on how to consider an arbitrarily high term in the perturbation theory . Only low order truncations were considered .

A way to overcome difficulty (B) was suggested by Belinicher and L'vov who introduced a novel transformation that allowed on one hand the elimination of the sweeping that leads to infrared divergences , and on the other hand allowed the development of simple rules for writing down any arbitrary order in the perturbation theory for the statistical quantities . The essential idea in this transformation is use of a coordinate frame in which velocities are measured relative to the velocity of one fluid particle . The use of this transformation allowed the examination of the structure functions of velocity differences $S_n(R)$ to all orders in perturbation theory . Of course , difficulty (A) remains ; the perturbation series still diverges rapidly for large values of Re , but now standard flied theoretic methods can be used to reformulate the perturbation expansion such that the viscosity is changed by an effective " eddie viscosity " . The theoretical tool that achieves this exchange is known in quantum field theory as the Dyson line resummation . The result of this procedure is that the effective expansion parameter of the order of unity . Of course , such a perturbation series may still diverge as a whole . Nonetheless it is crucial to examine first the order–by–order properties of series of this type .

Such an examination leads to a major surprise : every term in this perturbation theory remains finite when the energy–input scale L goes to ∞ and the viscous – dissipation scale n goes to 0 . The meaning of this is that the perturbative theory for S_n does not indicate the existence of any typical length-scale . Such a length is needed in order to represent deviations in the scaling exponents from the predictions of Kolmogorov's dimensional analysis in which both scales Land n are assumed irrelevant . In other areas of theoretical physics in which anomalous scaling has been found it is common that the perturbative series already indicates this phenomenon . In many cases this is seen in the appearance of logarithmic divergences that must be tamed by truncating the integrals at some renormalization length . Hydrodynamic turbulence seems at this point different . The nonlinear Belinicher-L'vov transformation changes the underlying linear theory such that the resulting perturbative scheme for the structure functions is finite order by order . The physical meaning of this result is that as much as can be seen from this perturbative series the main effects on the statistical quantities for velocity differences across a scale R come from activities on scales comparable to R . This is the perturbative justification of the Richardson-Kolmogorov cascade picture in which widely separated scales do not interact .

(1-2) Equation of Continuity

In the first instance we consider the mathematical ezepression of the principle of continuity. If we consider any closed surface in the fluid fixed in space, then in the absence of sources or sinks in the interior of the surface, the rate of increase of mass within the surface is equal to the rate at which mass flows into the volume enclosed by the surface. If (ρ) density of the fluid then the mass enclosed by the surface is $\int_v$ pdt, where (v) is the volume enclosed by the surface (S).

Now if (v) denotes the velocity of a particle of the fluid, its component in the direction (n) will be (V.n) so that the rate at which mass flows out of the volume (V) is $\int_s$ (V.n) pds, which , by Gauss' theorem , may be written as the volume integral $\int_v$ div (pv) dτ

The amount of fluid which flows per unit time into the volume will be this integral multiplied by -1 , so that we have the equation.

$$\frac{\partial}{\partial t}\int_{v} \rho d\tau = -\int_{v} div\ (\rho v) d\tau \dots\dots\dots\dots (1.2.1)$$

The volume (V) is arbitrary so that continuity condition is simply.

$$\frac{\partial \rho}{\partial t} + div\ (\rho v) = O \ \dots\dots\dots\dots\dots\dots\dots.(1.2.2)$$

Now, if (θ) is any physical quantity associated with the fluid $(\frac{\partial \theta}{\partial t})$ is the rate of change of (θ) measured at the fixed point r = (x, y, Z) and may be called the local rate of change of (θ).

In calculating the total rate of change of (θ) we must, however, include the rate of change arising from the fact that (θ) is being convicted by the fluid. A particle which is at the point (r) at time (t) is at the point (r+v∇t) at time $(t + \nabla t)$ so that the total change in (θ) is :

θ (r+v∇ t,t +∇t) - θ(r,t) . If we denote this change by : $\frac{D\theta}{Dt}\nabla t$

We then find, from Taylor's theorem for a function of two Variables, that :

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \ v.grad\ \theta \dots\dots\dots\dots\dots\dots(1.2.3)$$

The equation of continuity may be written in the form :

$$\frac{\partial \rho}{\partial t} + \rho\ div\ (v) + \ v.grad\ \rho = 0 \ \dots\dots\dots(1.2.4)$$

That is :

$$\frac{1}{\rho} + \frac{D\rho}{Dt} + \ div = 0 \dots\dots\dots\dots\dots\dots(1.2.5)$$

(1-3) Equation of motion

Consider a small element of volume $(\delta\tau)$ of the fluid and suppose there is an external force (F) per unit mass acting on the fluid, then if (δt) is taken as a particle its equation of motion can be written down, using Newton's second law of motion :

Force = Mass X Acceleration.

The force on ($\delta\tau$)will arise from the external force (PF $\delta\tau$) plus the force of viscosity M ($\partial\tau$) minus the resultant of the Pressure intensity integrated over the surface of the element. Summed for a large number of such elements comprising a volume (τ) contained within a surface S, the resultant external force will be ($\int_\tau PFd\tau$) and the resultant force of viscosity ($\int_\tau Md\tau$) .

The effects of the normal pressure over the surfaces of elements in contact with one another will cancel out and the total contribution of the fluid pressure to the resultant force will be – ($\int_s Pds$) . Hence :

$$\int_\tau \rho F d\tau + \int_\tau M dt - \int_s P ds = \int_\tau P \frac{dq}{dt}\, dt. \ldots\ldots (1.3.1)$$

Using an extension of Gauss' theorem we have

$$\int_s \rho ds = \int_\tau grad\ p d\tau \ \ldots\ldots (1.3.2)$$

And therefore :

$$\int_\tau \left\{ p\frac{dq}{d\tau} - \ PF - M + grad\ P \right\} d\tau = 0 \ldots. (1.3.3)$$

This will be true for any arbitrary volume (τ) of the fluid and therefore the integrand must vanish. Substituting for (M) , one form of the equation of motion for constant viscosity can be written :

$$P\,\frac{dq}{dt} = P\frac{\partial q}{\partial t} + P(\,q.\nabla)q$$
$$= PF - grad\ P + 2\,\mu\nabla^2 q + \mu curl\xi\ \ldots\ldots.\ \ (1.3.4)$$

If the fluid is incompressible then it follows that $\nabla^2 q = \ -curl\ \xi$ and the equation of motion can be written :

F- $\frac{1}{P}$ Grad P

$$\frac{dq}{dt} = \ F - \frac{1}{P}\ grad\ P + v\ \nabla^2 q\ \ \ldots\ldots..(1.3.5)$$

If the external force (F) is conservative it can be expressed as the gradient of ascalor function (Ω) ,

$$\text{F} = \text{-grad } \Omega \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ \ (1.3.6)$$

Then we have :

$$(q.\nabla)q = \ grad\frac{1}{2}\ \ \text{q}^2 - q\ \text{X}\ curL\ q\ \ldots\ldots\ldots.(1.3.7)$$

And

$$\frac{1}{\text{P}}\ \text{grad P} = \frac{\text{n}^\wedge}{\text{P}}\frac{\partial \text{P}}{\partial \text{n}} = \text{n}^\wedge\frac{\partial}{\partial \text{n}}\int\frac{1}{\text{p}}\frac{\partial \text{P}}{\partial \text{n}}\ \text{dn}$$
$$= \ \text{n}^\wedge\frac{\partial}{\partial \text{n}}\ \int\frac{\text{dP}}{\text{p}}\ = \text{grad}\ \int\frac{\text{dP}}{\text{p}}\ldots..(1.3.8)$$

Since (P) is in general a function of (P) ; here ($n^\wedge$) is a unit vector in the direction of grad (P) at the point under consideration and ($\frac{\partial}{\partial n}$) denotes differentiation in that direction. [2]

Hence the equation of motion of an in viscid fluid, putting (μ) = o can be written :

$$\frac{\partial P}{\partial t} - \ qX\ curLq = -grad\left(\frac{q^2}{2} + \int\frac{dP}{p}\ \ + \ \Omega\ \right);\ \ldots\ \ (1.3.9)$$

can also be written in a similar form. [2]

The equation of motion of an in viscid fluid can be derived directly in a similar way to omitting the force of viscosity from the argument. These are vector equation, the three Cartesian equations comprising are :

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \mathrm{v}\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial Z} = \mathrm{F_x} - \frac{1}{P}\,\frac{\partial P}{\partial X} + v\,\nabla^2 u \ldots \quad (1.3.10)$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial \mathrm{v}}{\partial X} + \mathrm{v}\,\frac{\partial \mathrm{v}}{\partial y} + w\,\frac{\partial \mathrm{v}}{\partial Z} = \mathrm{F_y} - \frac{1}{P}\,\frac{\partial P}{\partial y} + v\,\nabla^2 \mathrm{v} \ldots \quad (1.3.11)$$

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial X} + \mathrm{v}\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial Z} + \mathrm{F_Z}$$
$$- \frac{1}{P}\,\frac{\partial P}{\partial z} + v\,\nabla^2 w \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (1.3.12)$$

$$\text{Where}\quad \mathrm{F} = \mathrm{iF_X} + \mathrm{j\,F_y} + \mathrm{k\,F_Z} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.3.13)$$

The solving of problems is essentially to find a solution of both the appropriate equation of motion and the continuity equation which satisfies the given boundary conditions. [2]

(1-4) Vorticity Equation

The equation of motion for an incompressible fluid can be written in terms of the vorticity by substituting

$$\frac{\partial q}{\partial t} - q \times \xi = -grad\left(\frac{P}{\rho} + \frac{\mathrm{q}^2}{2} + \Omega\right) + v\,\nabla^2 q \ldots \ldots . \quad (1.4.1)$$

And taking the curl of both sides remembering that curl grad ≡ o

$$curL\;\frac{\partial q}{\partial t} - curL\,(q \times \xi) = vcurL\,(\nabla^2 q) \ldots\ldots \quad (1.4.2)$$

Curl is commutable with $(\frac{\partial}{\partial t})$ and (∇^2) .

$$\frac{d\xi}{dt} = \frac{\partial\xi}{\partial t} + (q.\nabla)\xi = (\xi.\nabla)q + v\,\nabla^2\xi \;\ldots\ldots\;\; (1.4.3)$$

Since div (ξ) = div curl q = 0 … … … … … … … … …. (1.4.4)

If the motion is slow (as might be the case just after it had been started from rest) the first term on the right is negligible and

$$\frac{\partial\xi}{\partial t} = v\,\nabla^2\xi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\;\; (1.4.5)$$

This equation is of the same form as the equation for the conduction of heat and by analogy vorticity cannot originate in the interior of the liquid but may be generated by spreading in wards from its boundaries. In fact it is noticed in actual fluids that vorticity exists to a marked extent only in those parts of the fluid which have passed near the bound arise. [2]

For an in viscid compressible fluid, an exactly similar we obtain Helmholtz, equations :

$$\frac{d}{dt}\left(\frac{\xi}{P}\right) = \frac{1}{P}\,(\xi.\nabla)q.\;\ldots\ldots\ldots\ldots\ldots\ldots.\;\; (1.4.6)$$

Examples :

1. If the fluid is not in motion – the hydrostatic case :

$$\frac{dq}{dt} = 0\,, \nabla^2 q = 0$$

Hence

$$\mathrm{F} = \mathrm{grad}\int\frac{\mathrm{dP}}{\mathrm{p}}$$

Therefore the system of forces must be conservative and holds hence.

$$dp = - pd\,\Omega$$

Therefore the surface (Ω) = constant must be the same as the surfaces P =constant, i.e., the surfaces of equal pressure and Potential coincide.

2. For in viscid liquid rotating about a vertical axis with constant angular velocity (ω) under gravity as if solid, if $(\hat{z})$ is a unit vector up wards and $(\hat{r})$ is a unit vector out wards from the axis, then :

$$F = -g\hat{Z} \;, \frac{dq}{dt} = -\omega^2 \acute{r}\, \mathrm{r}^{\wedge}$$

If (P) is taken as constant, becomes grad (P/$\mathcal{P}$) = - $gZ^{\wedge} + \omega^2\, \acute{r}\, \mathrm{r}^{\wedge}$

Hence :

$$p/\mathcal{P} = -gZ + \omega^2 r^2 + \text{constant}$$

If (P) is the pressure at the origin we have

(p – P) /$\mathcal{P}$ = -gZ + $\frac{1}{2}\,\omega^2 r^2$.

The curves of equal pressure, (P) = constant are Paraboloids of revolution ; in particular the surface of the liquid will take this shape.

This is easily demonstrated if a cup of tea is stirred and then left to spin – (g) has components (- w y , w x , o) and (ξ) components (o,o,2w) so that the motion is rotational and no velocity potential exists. [2]

Chapter Two

Chapter Two

perfect Fluid

(2-1) Irrotational Flow of a perfect Fluid

We shall begin our discussion of typical boundary value problems in hydrodynamics by considering the irrotational flow of a perfect fluid under certain circumstances.

In the case in which the motion of the fluid is irrotational the equation of motion assume aparticulary simple form . The motion of a fluid is said to be irrotational if the vorticity ω is Zero at every point of the fluid . In other words , curl v = 0 every where so that we may express the velocity vector v in terms of a scalar quantity Ø by means of the equation

$$v = -grad\ \emptyset \quad \text{...}(2.1.1)$$

A perfect fluid is one which is non viscous and which has constant density ρ . For such a fluid the equation of continuity (1) becomes simply

div v = 0 ……………………………………………………(2.1.2)

or , by virtue of equation (2.1.1)

$$\nabla^2 \emptyset = 0 \quad \text{..} (2.1.3)$$

Showing that the velocity potential Ø is a harmonic function . (1)

(2-2) Irrotationalality and potential flow

First, why should we care about irrotational flw at all? As ocean and atmosphere folks, the answer is probably that we shouldn't; vorticity is hugely important in stratified fluids on a rotating sphere. However, it is used extensively in most other fields, typically for looking at flow around objects (wings, for example). As such, it is important that you know what it is and how to take advantage of it (and it is probably directly useful for you ocean engineer types .

Consider $\omega = \nabla \times u = 0$ for irrotational flow. Recall from *problem set 3 that*

$$\nabla \cdot \omega = \nabla \cdot (\nabla \times u) = 0 \quad \text{....................(2.2.1)}$$

More generally, the divergence of the curl of any vector function is zero where the mixed partials are equal. Well, it is also easy to show that the curl of the gradient of a scalar function is zero,that is

$$\nabla \times (\nabla\varphi) = 0 \quad \text{..(2.2.2)}$$

Again, this assumes the equality of the mixed partials. Try it!

So, if we know $\nabla \times u = 0$, and we know $\nabla \times (\nabla\varphi) = 0$, then $u = \nabla\emptyset$,

$$u = \frac{\partial\emptyset}{\partial x} \;, v = \frac{\partial\emptyset}{\partial y} \;, w = \frac{\partial\emptyset}{\partial z} \;\ldots\ldots\ldots(2.2.3)$$

∂Where ϕ is the velocity potential. This holds for any irrotational flow (in fact, the existence of ϕ is the sole criterion of irrotationality) and tells us that the three velocity components can be derived from a single variable, ϕ(x, y, z, t). Irrotational flow is called potential flow. Thus, when we have an irrotational Incompressible fluid ,

$$\nabla . u = 0 \;\Rightarrow\; \nabla . \nabla\varphi = 0 \;\Rightarrow\; \nabla\varphi = 0 \ldots . . (2.2.4)$$

Therefore, the velocity potential obeys Laplace's equation. Laplace's equation is a classic problem in intro PDE classes. You might recall solving Laplace's equation on various geometries using separation of variables.

This $u = \nabla\varphi$ stuff looks a little like the relationship between velocity and streamfunction. remember the streamfunction? Remember the streamfunction from kinematics? If we have an incompressible flow, then

$$\nabla \cdot u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2.4)$$

We know we can satisfy this relationship be defining

$$u = \nabla \times a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2.5)$$

since $\nabla \cdot (\nabla \times a) = 0$. (Recall from before how the divergence of the curl of a vector function is zero). Because a is a vector, there is generally little to be gained taking this approach (u is a vector as well). But, if we have a 2D flow, then both the ***x***-component and *y*-component of a are 0.

$$a = 0\,i + 0\,\mathbf{j} + \psi\,k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2.6)$$

$$\Rightarrow \nabla \times a = \frac{\partial\psi}{\partial y} i \; \frac{\partial\psi}{\partial x} j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2.7)$$

$$\Rightarrow \quad \nabla.(\nabla \times a) = \frac{\partial^2\psi}{\partial y \partial x} - \frac{\partial^2\psi}{\partial x \partial y} = 0 \;\; \ldots (2.2.8)$$

So

$$u = \frac{\partial\psi}{\partial y} \quad , \quad v = \frac{\partial\psi}{\partial x} \ldots\ldots\ldots\ldots\ldots (2.2.9)$$

Where $\psi \equiv$ streamfunction. ψ is constant along a streamline. A check of this starts with

$$d\psi = \frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial y} dy$$

$$= -vdx + udy \ldots\ldots\ldots\ldots\ldots (2.2.10)$$

We know that on a streamline

$$\frac{dx}{u} = \frac{dy}{v} \;\; or \;\; \frac{dy}{dx} = \frac{v}{u} \;\; \ldots\ldots\ldots\ldots\ldots..(2.2.11)$$

which can only be satisfied if $d\psi = 0 \Rightarrow \psi =$ constant.Thus in 2D a stream function can be defined when $\nabla\cdot$ u = 0. If the flow is irrotational, that is, $\nabla\times$ u = 0, and if $\nabla\cdot$ u = 0, this $\nabla^2 \phi = 0$. (Jim – This is what it looked like you were trying t say in your notes; is this how you want it?)

Laplace's equations

If there is a flow that is 2D, incompressible, and irrotational, then

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\emptyset}{\partial x} \quad , \qquad v = -\frac{\partial\psi}{\partial x} = \frac{\partial\emptyset}{\partial y} \ldots\ldots(2.2.12)$$

Note that ψ and φ are orthogonal. Thus,

$$\nabla.\mathrm{u} = 0 \;\Rightarrow\; \nabla^2\varphi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.2.13)$$

Again, these are the Laplace's equations. Laplace's equation represents, or should represent a familiar problem for you. The Dirichlet problem for the Laplace equation goes like

$$\nabla\varphi = a, \qquad \varphi = \varphi o \ldots\ldots\ldots\ldots.(2.2.14)$$

on boundary (2.2.14) The Neumann problem goes like

$\nabla\varphi = a, \quad \varphi^2 = \varphi o$ normal to boundary on boundary

They are well studied problems, with applications beyond fluids, and have well-behaved, stable, unique solutions. Again, in ocean and atmospheres we have little call for potential flow, but it is important you lot know it exists.

Aside on velocity potential

Why is it defined u = $\nabla\varphi$? This means that u moves up gradient! Why is it not defined u = − $\nabla\varphi$? Mathematically, this would be perfectly fine. To explain the sign convention, we need to bring in the stream function as well. We define it as

$$u = \frac{\partial\psi}{\partial y} \ , \ v = -\frac{\partial\psi}{\partial x} \ldots\ldots\ldots..(2.2.15)$$

This is another arbitrary sign choice. We could have said

$$u = -\frac{\partial\psi}{\partial y} \ , \ v = \frac{\partial\psi}{\partial x} \ \ldots\ldots\ldots\ldots..(2.2.16)$$

In fact, GFD folks tend to use this definition – more on this later. It turns out that these two sign choices are related. Recall Greg telling you about the Cauchy-Riemann conditions?

$$\frac{\partial\emptyset}{\partial x} = \frac{\partial\psi}{\partial y} \ , \qquad \frac{\partial\emptyset}{\partial y} = -\frac{\partial\psi}{\partial x} \ldots\ldots\ldots\ldots.(2.2.17)$$

These equations allow us to write w = ϕ + iψ (the complex potential) which gives

$\frac{d\omega}{dz} = u - iv$ (the complex velocity)

and permits one to use all the tools of complex functions to solve 2D fluids problems. If we are to define (in accordance to Kundu)

$$u = \frac{\partial\emptyset}{\partial x} i + \frac{\partial\emptyset}{\partial y} j = \nabla\emptyset \ \ \ldots\ldots\ldots\ldots\ldots(2.2.18)$$

$$u = \frac{\partial\psi}{\partial y} i - \frac{\partial\psi}{\partial x} j \ \ \ldots\ldots\ldots\ldots\ldots\ldots.\ \ (2.2.19)$$

then the Cauch-Riemann conditions are satisfied as we have

With the potential flow defined u $= -\nabla$ and u = $\frac{\partial\psi}{\partial y}i + \frac{\partial\psi}{\partial x}j$,

the high and low ψ and p contours are in matching locations. Rather, if we define (in accordance with Lamb)

$$u = \frac{\partial\Phi}{\partial x}i - \frac{\partial\Phi}{\partial y}j = -\nabla\Phi \ldots. (2.2.20)$$

and

$$u = -\frac{\partial\psi}{\partial y}i + \frac{\partial\psi}{\partial x}j \ldots\ldots (2.2.21)$$

then the Cauch-Riemann conditions are still satisfied

$$-\frac{\partial\emptyset}{\partial x} = -\frac{\partial\psi}{\partial y}\ , \qquad \frac{\partial\emptyset}{\partial y} = \frac{\partial\psi}{\partial x} \ldots\ldots\ldots (2.2.22)$$

$$\Rightarrow \frac{\partial\emptyset}{\partial x} = \frac{\partial\psi}{\partial y}\ , \ \frac{\partial\emptyset}{\partial y} = -\frac{\partial\psi}{\partial x} \ldots\ldots.. (2.2.23)$$

Thus, in a sense, both definitions are "right" in that they both work. In fact, a mixed signs version could also be defined, but this is inconvenient from the Cauchy-Riemann perspective. So which is "better"? Probably

$$u = -\nabla\varphi\ , \qquad u - \frac{\partial\psi}{\partial y}i + \frac{\partial\psi}{\partial x}j \ldots\ldots\ldots (2.2.24)$$

Why?

a) Velocity flows down gradient.

b) Intuitive relatioship between ψ and p in geostrophic flow .

This explains the conventions in GFD books (although most never discuss u $= -\nabla\varphi$). (Jim – This does not make sense to me. To you mean to say that most never discuss u $= \nabla\varphi$?). Why does Kundu go the other way? Probably because Bachelor does it that way. Why does Bachelor do it that way? He derives u $= \nabla\varphi$ and states "there is no question here of an interpretation of φ as a potential energy function". Presumably this is because he invoked only maths to obtain u $= \nabla\varphi$, and there is no potential energy invoked. (Jim – Hmmm... I do not think that "maths" is a word).

Bernoulli form of the Euler equations

Back to the Bernoulli form of the Euler equations

$$\frac{\partial u}{\partial t} + \nabla B = u \times omega \ \ldots\ldots\ldots\ldots..(2.2.25)$$

If we assume irrotational flow, then u $\times\, \omega = 0$, but we also have the opportunity to write $\frac{\partial \mathbf{u}}{}$ in the form of a gradient using the velocity potential. For irrotational flow, u $= \nabla\varphi$, and we have

$$\nabla\frac{\partial \emptyset}{\partial t} + \nabla \mathrm{B} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2.26)$$

Classic example of irrotational, laminar flow analyzed using the Bernoulli function. As is seen, there is a pipe with a section with a small diameter relative to either side of it . Following a streamline through the center of the pipe,

$$\rho\frac{1}{2}u_1^2 + p_1 = \rho\frac{1}{2}u_2^2 + p_2 = \rho\frac{1}{2}u_3^2 + p_3. \ldots\ldots.(2.2.27)$$

By inspection, $u1 = u3 < u2$. For the Bernoulli function to be constant at each point, $p1 = p3 > p2$.

The flow around a wing can approximately be modeled as steady and irrotational. This implies that the Bernoulli function is constant everywhere. Thus, for the three points that are indicated on that

$$\rho\frac{1}{2}u_1^2 + p_1 = \rho\frac{1}{2}u_2^2 + p_2 = \rho\frac{1}{2}u_3^2 + p_3 \ldots.(2.2.28)$$

assuming that the $\rho g \Delta z$ terms are small. Since the air over the top of the wing has a longer distance to travel, $u1 = u3 < u2$ and hence $p1 = p3 > p2$. Since $p3 > p2$, the lower pressure above the wing acts to "suck" of "hold" the wing up. (Jim – Is this really how it should be modeled? The impression that I got from the spirited discussion that ensued from this example last year was that this was a really poor description of the system.)

Integrate out the space derivatives, and you get

$$\frac{\partial \emptyset}{\partial \mathrm{t}} + \frac{1}{2} u.u + gz + \int \frac{dp}{\rho} = f(t) \ldots.(2.2.29)$$

This is Bernoulli's equation for irrotational flow, where $\boldsymbol{f}$ is spatially uniform at each t. This form of Bernoulli's equation is used to solve the gravity wave problem; pretty much the only problem in GFD that is irrotational.

Let's stay with irrotational Bernoulli, and look at an example in steady state (laminar, not turbulent). Further, let's assume $\rho =$ constant, so

$$\frac{1}{2} u.u + gz + \frac{p}{\rho} = constant \;\ldots\ldots(2.2.30)$$

Note: Bernoulli is really an expression of conservation of energy where the first term is kinetic, the second potential, and the third the energy change due to temperature or volume changes. It is best to think of Bernoulli as a kinematic expression here. (Kinematic, not dynamic; explains, but does not predict). If we know something about the velocity distribution, then we know something about the pressure distribution. Once we know the pressure distribution, we can say something about the forces on, for example, the wing. Similarly, if we know the pressure distribution, Bernoulli lets us say something about the velocity distribution.

Another example of the application of Bernoulli is the Pitot tube as seen in figure 10.5. Here again the flow is steady and irrotational, and ρ is constant. Hence, from comparing points 1 and

Diagram of Pitot tube configuration. 2 on the diagram, we have that

$$\rho\frac{1}{2}\mathrm{u}_1^2 + p_1 = \rho\frac{1}{2}\mathrm{u}_2^2 + \mathrm{p}_2 \ldots\ldots\ldots.(2.2.31)$$

At point 2, $u2 = 0$, so

$$\rho\frac{1}{2}u_1^2 + p_1 = p_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots.(2.2.32)$$

$$u_1 = \sqrt{\frac{2(\,p_2 - p_1\,)}{\rho}} \;\ldots\ldots\ldots\ldots\ldots..(2.2.33)$$

Note that this is exactly how airplanes determine their air speed. In this example, we can use the hydrostatic relation to get $p1$ and $p2$, in terms of Δz.

$$p_1 = \rho g z_1 \;\;,\;\; p_2 = \rho g z_2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2.2.34)$$

$$u_1 = \sqrt{2(gz_2 - gz_1)} = \sqrt{2g\Delta z} \;\ldots\ldots\ldots..(2.2.35)$$

Note that we can only do this when all streamlines at 1 are parallel between 1 and the upper wall.
There are a bunch of different pressures here:

$\rho\frac{1}{2}u^2 + p \;\; \equiv$ stagnation pressure
$\rho\frac{1}{2}u^2 \equiv$ dynamic pressure
$p \equiv$ static pressure

(2-3) Two – dimensional flow

In the first instance we shall consider the irrotational two dimensional flow of a perfect fluid filling the half space ($y \geq 0$) .

It follows from this equation :

$$V = -grad\; \emptyset \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3.1)$$

and = $\nabla^2\emptyset = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.3.2)$

That in terms of a velocity potential (θ) , we have :

$$u = -\frac{\partial\emptyset}{\partial x} \;\;,\;\; V = -\frac{\partial\emptyset}{\partial y} \;\ldots\ldots\ldots..\;\;(2.3.3)$$

Where the scalar velocity potential (∅) satisfies the two – dimensional form of Laplace's equation

$$\nabla^2 \emptyset \equiv \frac{\partial^2 \emptyset}{\partial \varkappa^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0 \; \dots \dots \dots .. \; (2.3.4)$$

We shall suppose that fluid is introduced to the half space through the strip ($/\varkappa/ \leq a$) of the plane (y = 0) . first we shall assume that the fluid is introduced normally with prescribed velocity so that, along (y = o) we have the boundary condition.

$$\frac{\partial \emptyset}{\partial \mathrm{y}} = \begin{Bmatrix} -f(\varkappa) & 0 </\varkappa/< a \\ 0 & /\varkappa/ > 0 \end{Bmatrix} \dots \dots \dots .. (2.3.5)$$

Where f ($\varkappa$) is a prescribed function of position . we further make the assumption that at a great distance from the plane (y = o) the fluid is at rest, that is

$$(u, \mathrm{v}) \to 0 \quad as \;\; y \to \infty \quad \dots \dots \dots .... \;\; (2.3.6)$$

The Laplace equation for velocity potential function (∅) is valid in both two and three dimensions and in any coordinate system but only in irrotational regions of flow.

The laplacian operator (∇^2) is a scalar operator defined as ($\vec{\nabla} . \vec{\nabla}$) , and is called Laplace equation. [4]

If , now we introduce the Fourier transforms.

$$\emptyset(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi\varkappa} \, \emptyset(\varkappa, y) d\varkappa$$

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi\varkappa} \, F(\varkappa) d\varkappa$$

We find that equation (2.3.4) and (2.3.5) are equivalent to the ordinary differential equation :

$$\frac{d^2\emptyset}{dy^2} - \xi^2\emptyset = 0$$

Taken with the boundary condition $(\frac{d\emptyset(\xi,0)}{dy}) = -F(\xi)$.

The solution of this problem subject to the condition (2.3.6) is obviously :

$$\emptyset = \frac{F(\xi)}{/\xi/} \, e^{-/\xi/y}$$

Whence, by the inversion theorem (10)

$$\emptyset = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(\xi)}{/\xi/} \, e^{i\xi\varkappa - /\xi/y} \, d\xi \; \ldots\ldots\ldots\ldots\ldots. \quad (2.3.7)$$

In Particular : if

$$F(\varkappa) = \begin{cases} u, & 0 </\varkappa/< a \\ 0, & /\varkappa/ > 0 \end{cases}$$

We have :

$$F(\xi) = \frac{\text{U}}{\sqrt{2\pi}} \; \frac{\text{Sin}(\xi\text{a})}{\xi}$$

So that :

$$\emptyset = \frac{U}{2\pi} \int_{-\infty}^{\infty} \frac{\text{Sin}(\xi\text{a})}{\xi} \; \frac{e^{i\xi\varkappa}}{/\xi/} \; e^{-/\xi/y} \, d\xi$$

and the component the fluid velocity in the (y) direction is given by :

$$\text{v} = -\frac{\partial\emptyset}{\partial\text{y}} = \frac{\text{U}}{2\pi} \int_{-\infty}^{\infty} \frac{\text{Sin}(\xi\text{a})}{\xi} \; \text{e}^{-/\xi/\text{y}+\text{i}\xi\varkappa} \, \text{d}\xi$$

Making use of the result

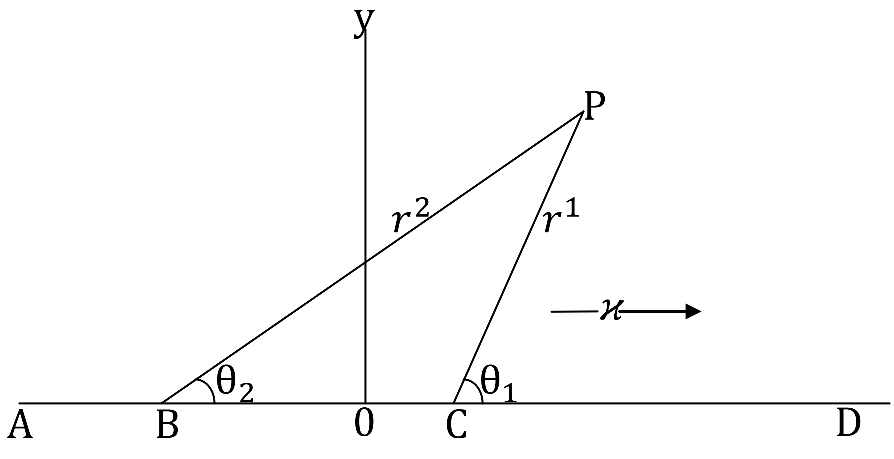
$$\int_{-\infty}^{\infty} \frac{\text{Sin}(\propto \xi)}{\xi} \; \text{e}^{-\xi\text{y}} \, \text{d}\xi = \frac{\pi}{2} - \tan^{-1}\frac{\text{y}}{\propto}$$

We find that

$$\text{v} = \frac{\text{U}}{2\pi} \, (\theta_1 - \theta_2)$$

Where $\tan \theta_1 = y / (\varkappa - a)$ and

$\text{Tan} \; \theta_2 = y / (\varkappa + a)$



(Fig 1)

Geometrical interpretation of

r_1 , r_2 , θ_1 and θ_2 . (C) is the point

(a,0) and B (-a,0). [1]

In a similar fashion we find for the component of the fluid velocity in the ($\varkappa$) direction

$$u = -\frac{\partial\emptyset}{\partial\varkappa} = -\frac{iU}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Sin}(\xi a)}{/\xi/}\ e^{i\xi\varkappa-/\xi/y}\,d\xi = \frac{U}{2\pi}\ \mathrm{Log}\ \frac{r_2}{r_1}$$

Where

$$r_2^2 = (\varkappa + a\,)^2 + y^2\ and\ r_1^2 = (\varkappa - a\,)^2 + y^2$$

If we introduce a complex Potential

W = $\emptyset + i\psi$, we have :

$$\frac{d\omega}{dZ} = \frac{\partial\emptyset}{d\varkappa} - i\,\frac{\partial\emptyset}{dy} = -u + iv \ \ldots\ldots\ldots\ldots \quad (2.3.8)$$

So that , inserting the above values for the components of velocity, we obtain the expression

$$\frac{d\omega}{dZ} = -\frac{U}{2\pi}\left[\text{Log}\frac{r_1}{r_2} + i\,(\theta_1 - \theta_2)\right] = \frac{U}{2\pi}\,Log\,\frac{Z-a}{Z+a}$$

Integrating this expression with respect to (Z),we obtain for the complex potential :

$$w = -\frac{U}{2\pi}\left[2a + (Z-a)Log\,(Z-a) - (Z+a)Log\,(Z+a)\right] \quad (2.3.9).$$

(2-4) Steady flow of a perfect fluid through a slit

We shall next consider the problem of determining the two-dimensional steady flow of a perfect fluid through a slit in plane rigid boundary. With the center of the slit as origin, and with they axis perpendicular to the plane of the thin screen, we have to solve the differential equation to the equation :

$$\nabla_1^2\emptyset \equiv \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} = 0$$

Subject to the boundary conditions.

$$\emptyset = const\,, /\varkappa/ \le a\,,\ \ v = \frac{\partial\emptyset}{\partial y} = 0$$

$$/\varkappa/ > a \text{ when } y = 0$$

If along y = o we have :

$$v = \begin{cases} (a^2 - \varkappa^2)^{-\frac{1}{2}}, & 0 < /\varkappa/ < a \\ 0, & /\varkappa/ > 0 \end{cases}$$

Then by the analysis of two-dimensional flow we find that the velocity potential is given by equation :

$$\emptyset = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(\xi)}{/\xi/} \; e^{i\xi\varkappa - /\xi/y} \; d\xi$$

with :

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{e^{-i\xi\varkappa}}{(a^2 - \varkappa^2)^{\frac{1}{2}}} \, d\varkappa = \sqrt{\frac{2}{\pi}} \int_{0}^{a} \frac{\cos(\xi\varkappa)}{(a^2 - \varkappa^2)^{\frac{1}{2}}} \, d\varkappa = \sqrt{\frac{\pi}{2}} \, J_0(a\xi).$$

Hence we have for the velocity potential

$$\emptyset = \frac{1}{2} \int_{-\infty}^{\infty} \frac{J_0(a\xi)}{/\xi/} \; e^{i\xi\varkappa - /\xi/y} \; d\xi$$

So that from equations :

$$u = -\frac{\partial \emptyset}{\partial \varkappa} \quad , \quad v = \frac{\partial \emptyset}{\partial y}$$

We have for the components of the velocity vector :

$$v = -\frac{\partial \emptyset}{\partial y} = \int_{-\infty}^{\infty} e^{-\xi y} \, J_0(a\xi) \cos(\xi\varkappa) \; d\xi \; \ldots \ldots (2.4.1)$$

$$u = -\frac{\partial \emptyset}{\partial \varkappa} = \int_{0}^{\infty} e^{-\xi y} \, J_0(a\xi) \sin(\xi\varkappa) \; d\xi \; \ldots \ldots (2.4.2)$$

Also , when y = o

$$\frac{\partial \emptyset}{\partial \varkappa} = -\int_{0}^{\infty} \sin(\xi\varkappa) \; J_0(a\xi) \, d\xi = 0$$

If $/\varkappa/ < a$

verifying that (Ø) is a constant on the segment y = o , $(/\varkappa/< a)$. [1]

Substituting from equations (2.4.1) and (2.4.2) into equation (2.4.8) , we find that the complex potential of the flow is given by the solution of the equation :

$$\frac{d\omega}{dZ} = i\int_0^\infty e^{i\xi z}\, \mathrm{J}_0(\mathrm{a}\xi)\, d\xi = (Z^2 - \mathrm{a}^2)^{-\frac{1}{2}}$$

Integrating this equation , we obtain the standard solution [1]

Z = a cosh (ω)

(2.5) flow of a jet of perfect fluid through a circular aperture in a plane rigid screen.

The solution to the problem of determining the steady flow of a perfect fluid through a circular aperture in a plane rigid wall is given on Lamb's treatise. We shall now show how this three – dimensional analogue of the problem considered in (2. 3) may be generalized by the use of the theory of dual integral equations developed in equation : [1]

$$\frac{\partial \mathrm{v}}{\partial \mathrm{t}} - 2\mathrm{v} \times \omega + \mathrm{grad}\left(\Omega + \frac{1}{2}\mathrm{v}^2 + \frac{\mathrm{P}}{\rho}\right) + 2\mathrm{v}\,\mathrm{curl}\omega = 0$$

With the center of the aperture as aperture as origin and with the (Z) axis perpendicular to the plane of the thin rigid screen we may describe any point in the fluid by means of cylindrical polar coordinates (r) and (Z).

The solution of the steady flow problem reguives the determination of a velocity potential function Ø(r,Z) satisfying Laplace's equation : $\nabla^2 Ø = 0$

In these coordinates ,

$$\frac{\partial^2 \emptyset}{\partial r^2} + \frac{1}{\mathrm{r}} \frac{\partial \emptyset}{\partial \mathrm{r}} + \frac{\partial^2 \emptyset}{\partial Z^2} = 0 \ldots \ldots \ldots (2.5.1)$$

Together with the boundary conditions

$$\emptyset = g(r) \qquad r < a \ldots \ldots \ldots \ldots \ldots \ldots (2.5.2)$$

$$\frac{\partial \emptyset}{\partial Z} = 0 \qquad r > a \ldots \ldots \ldots \ldots \ldots \ldots (2.5.3)$$

Over the plane Z = o . The function g(r) occurring in equation (2.5.2) is prescribed. [1]

In the special case considered by Lamb it is a constant.

Multiplying both sides of equation (2.5.1) by $rJo(\xi r)$ and integrating with respect to (r) from (O) to ∞, we find that this equation is equivalent to the second – order ordinary differential equation.

$$\frac{\mathrm{d}^2 \emptyset}{\mathrm{d}Z^2} = \xi^2 \emptyset = 0 \ldots \ldots \ldots \ldots \ldots (2.5.4)$$

for the determination of the Hankel transform.

$$\emptyset(\xi, \mathrm{Z}) = \int_0^\infty r\, \emptyset(r, Z)\, J_0\, (\xi\, r) dr \ldots \ldots (2.5.5)$$

of the velocity potential . If we are interested in flow into the half space Z≥ 0, then , since the velocity potential must tend to Zero as Z→ ∞ , we must take a solution of equation (2.5.4) of the form [1]

$$\emptyset = A\, (\xi)\, e^{-\xi Z} \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2.5.6)$$

Where A (ξ) is to be determined by the conditions (2.5.2) and (2.5.3).

Differentiating equation (2.5.6) with respect to (Z) , we have.

$$\int_0^\infty r \frac{\partial \emptyset}{\partial Z} J_0 (\xi r) dr = -\xi A(\xi)\ e^{-\xi Z}$$

and inverting these equations by means of the Hankel inversion theorem we obtain the expressions :

$$\emptyset = \int_0^\infty \xi A(\xi)\ e^{-\xi Z} r\ J_0 (\xi r) d\xi$$

$$\frac{\partial \emptyset}{\partial Z} = -\int_0^\infty \xi^2 A(\xi)\ e^{-\xi Z}\ J_0 (\xi r) d\xi$$

If we substitute from these equations into equations (2.5.2) and (2.5.3) and write $= r/a, F(u) = uA\,(u/a)\,, G(\rho) = a^2 g(\,r\,)$, we obtain the dual integral equations.

$$\int_0^\infty \mathrm{F(u)}\, J_0\ (\rho \mathrm{u}) \mathrm{du} = \mathrm{G}(\rho) \qquad 0 < \rho < 1$$

$$\int_0^\infty \mathrm{u\ F(u)}\, J_0\ (\rho \mathrm{u}) \mathrm{du} = 0 \qquad \rho > 1$$

for the determination of the function F(u) from which we derive the value of A(ξ) by the equation

$$A(\xi) = \frac{F(a\xi)}{a\xi}$$

The dual integral equations (2-4-7)

$$F(u) = \frac{2}{\pi} \cos u \int_0^1 \frac{\mathrm{yG(y)dy}}{(1-\mathrm{y}^2)^{\frac{1}{2}}} + \frac{2}{\pi} \int_0^1 \frac{\mathrm{ydy}}{(1-\mathrm{y}^2)^{\frac{1}{2}}}$$

$$\int_0^1 \mathrm{G(yu)} \varkappa \mathrm{u}\ \ \sin(\varkappa \mathrm{u})\, \mathrm{du} \qquad \ldots\ldots\ldots.. \ (2.5.8)$$

In the particular case in which the function G(ρ) is a constant, C, say, we find that

$$F(u) = \frac{2c}{\pi}\left(\frac{sinu}{u}\right) \ldots\ldots\ldots\ldots..(2.5.9)$$

Thus g(r) is constant (r) where C = $a^2 r$ and

$A(\xi) = \frac{2r}{\pi} \ \frac{\sin(\xi a)}{\xi Z}$ so that finally

$$\emptyset = \frac{2r}{\pi} \int_0^\infty \frac{\sin(\xi a)}{\xi} \ e^{-\xi Z} J_0 (\xi \, r) d\xi \quad \ldots\ldots \quad (2.5.10)$$

which is the solution derived other wise by Lamb. [1]

We can perform a similar analysis in the case in which $(\partial\emptyset / \partial Z)$ is prescribed all along the plane Z = o . If $\partial\emptyset / \partial Z$ = -F(r) (which is assumed to be Zero when (r) exceeds a) , and if we denote its Hankel transform of Zero order by :

$$\bar{F}(\xi) = \int_0^a \mathrm{r}\ \mathrm{F(r)}\, J_0\ (\xi_{\mathrm{r}})\ \mathrm{dr}$$

than it is readily shown that

$$\bar{\emptyset} = -\frac{F(\xi)}{\xi}$$

And so that

$$\emptyset = -\int_0^\infty \bar{F}(\xi)\ e^{-\xi Z} J_0 (\xi r)\, d\xi$$

If the aperture is very small, we may take

$$F(r) = \frac{S}{2\pi \mathrm{r}}\ \partial(r)$$

Of which the Hankel transform is

$\bar{F}$ (ξ) = S/2π , giving

$$\emptyset\,(r,Z) = -\frac{S}{2\pi}\int_0^\infty e^{-\xi Z}\,J_0\,(\xi r)\,d\xi = -\frac{S}{2\pi(r^2+Z^2)^{\frac{1}{2}}}$$

Similary , if we take $\mathrm{F(r)} = (a^2 - r^2)^{-\frac{1}{2}}$, we arrive at the solution. (2.5.10) .[1]

Chapter Three

Chapter Three

Waves

(3-1) Surface Waves

We have now to investigate, as far as possible the laws of wave motion in liquids when the vertical acceleration is no longer neglected. [6]

The most important case not covered by the preceding theory is that of waves on relatively deep water, where, as will be seen, the agitation rapidly diminishes in amplitude as we pass downwards from the surface ; but it will be understood that there is continuous transition to the state of things investigated in the preceding chapter, where the horizontal motion of the fluid was sensibly the same from top to bottom. [6]

We begin with the oscillations of a horizontal sheet of water, and we will confine ourselves in the first instance to cases where the motion is in two-dimensions, of which one ($\varkappa$) is horizontal, and the other (y) vertical. The elevations and depressions of the free surface will then present the appearance of a series of parallel straight ridges and furrows, perpendicular to the plane ($\varkappa y$).

The motion, being assumed to have been generated originally from rest by the action of ordinary forces, will necessarily be irriotational and the velocity Potential (∅) will satisfy the equation :

$$\frac{\partial^2 \emptyset}{\partial \varkappa^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0 \quad \dots\dots\dots\dots \; (3.3.1)$$

With the condition

$$\frac{\partial \emptyset}{\partial n} = 0 \quad \dots\dots\dots\dots.. \ (3.1.2)$$

at affixed boundary. [6]

To find the condition which must be satisfied at the free surface (P = const.) , let the origin (O) be taken at the undisturbed level, and let oy be drawn vertically upwards.

The motion being assumed to be infinitely small, we find, putting (Ω) = gy in the formula (u) and neglecting the square of the velocity (q),

$$\frac{P}{\rho} = \frac{\partial \emptyset}{\partial t} - gy + F(t) \quad \dots\dots\dots \ (3.1.3)$$

Hence if (η) denote the elevation of the surface at time t above the point ($\varkappa, 0$) , we shall have, since the pressure there is uniform,

$$\eta = \frac{1}{g}\left[\frac{\partial \emptyset}{\partial t}\right]_{y=\eta} \quad \dots\dots\dots\dots. \ (3.1.4)$$

Provided the function F(t), and the additive constant, be supposed merged in the value of ($\partial \emptyset / \partial t$) .

Subject to an error of the order already neglected, this may be written.

$$\eta = \frac{1}{g}\left[\frac{\partial \emptyset}{\partial t}\right]_{y=0} \quad \dots\dots\dots\dots\dots \ (3.1.5)$$

Since the normal to the free surface makes an infinitely small angle ($\partial\eta / \partial\varkappa$) with the vertical, the condition that the normal component of the fluid velocity at the free surface must be equal to the normal velocity of the surface itself gives, with sufficient approximation,

$$\frac{\partial\eta}{\partial t} = -\left[\frac{\partial\emptyset}{\partial t}\right]_{y=0} \quad \dots\dots\dots\dots\dots (3.1.6)$$

This is in fact what the general surface condition becomes, if we put F($\varkappa$,y,zt) ≡ y − η, and neglect small quantities of the second order. [6]

Eliminating (η) between (3.1.5) and (3.1.6) we obtain the condition

$$\frac{\partial^2\emptyset}{\partial t^2} + g\frac{\partial\emptyset}{\partial y} = 0 \quad \dots\dots\dots\dots (3.1.7)$$

to be satisfied when y = o . This is equivalent to DP/Dt = o. [6]

In the case of simple- harmonic motion the time-factor being ($e^{i(\sigma t+\varepsilon)}$) , this condition becomes

$$\sigma^2\emptyset = g\frac{\partial\emptyset}{\partial y} \quad \dots\dots\dots\dots\dots (3.1.8)$$

(3-2) Surface waves generated by an impulsive pressure

In the first boundary value problem of this type which we shall consider we suppose that the fluid is of infinite depth, y≤ o , and that waves are generated by the action of an impulsive pressure on the surface y = o of the fluid.To determine the wave system so produced we have, therefore, to solve equations

$$\frac{\partial^2 \emptyset}{\partial \varkappa^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0 \quad and\ equation : \frac{\partial^2 \emptyset}{\partial \varkappa^2} + g\frac{\partial \emptyset}{\partial y} = 0$$

subject to the boundary condition ($\emptyset = \frac{Po(\varkappa)}{P}$)

together with the initial condition that ($\eta = 0$)

when t = o.

If we introduce the Fourier transform

$$\Phi(\zeta, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \emptyset(\varkappa, y, t)\ e^{i\zeta Z} dZ$$

then the equation = ($\frac{\partial^2 \emptyset}{\partial \varkappa^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0$) is equivalent to

($\frac{\partial^2 \Phi}{\partial y^2} - \zeta^2 \Phi$) = 0

of which the solution which tends to zero as $Y \to -\infty$ is

Φ (ζ,t) = A (ζ,t) $e^{/\zeta/y}$ … … .. (3.2.1)

Multiplying both sides of equation ($\frac{\partial^2 \emptyset}{\partial t^2} + g\frac{\partial \emptyset}{\partial y} = 0$) by $(2\pi)^{-\frac{1}{2}} e^{i\zeta\varkappa}$ and integrating over the entire range of variation of x, we find that

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} + g/\zeta/A(\zeta, t) = 0$$

whence it follows that

$$A(\zeta, t) = \propto (\zeta) e^{i(\frac{g}{\zeta})t^{1/2}} + \beta\ (\zeta)_e i(\frac{g}{\zeta})t^{1/2} \to (\ 3.2.2)$$

where ($\propto(\zeta)$) and B(ζ) are constants of integration.

Along y = o we have

$$\frac{dI}{dt} = i(g/\zeta/)^{\frac{1}{2}}\left[\propto(\zeta)e^{i(g/\zeta/)\frac{1}{2}t} - \beta(\zeta)\, e^{-i(g/\zeta/)\frac{1}{2}t}\right]$$

so that if (η) is zero initially we must take $\propto(\zeta)$) = B(ζ) . Using this condition with that on (∅) , which may be written in the form I(ζ,o,o) = $\bar{P}_0(\zeta)$/p, we find that

$$\Phi = \frac{P_0(\zeta)}{\rho}\cos(g/\zeta/)^{\frac{1}{2}}te^{/\zeta/y}$$

which by the application of the inversion leads to the result

$$\emptyset(\varkappa, y, t) = \frac{1}{\rho}\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} \bar{P}_0(\zeta)\cos(g/\zeta/)^{\frac{1}{2}}te^{/\zeta/y-i\zeta Z}\,d\zeta$$

$$= \frac{1}{\pi\rho}\int_0^{\infty} d\zeta \int_{-\infty}^{\infty} P_0(\propto)\cos\left[(g|y|)^{\frac{1}{2}}t\right]\cos[(\varkappa-\propto)]e^{\zeta y}\, d\propto \;\ldots\ldots$$

$$\ldots\ldots\ldots(3.2.3)$$

In particular, if $\bar{P}_0(\zeta)$in an even function of (ζ) the first of these two equations may be put into the form.[1]

$$\emptyset(\varkappa, y, t) = \frac{1}{\rho\sqrt{2\pi}}\left[\int_0^{\infty} e^{\zeta y}\bar{P}_0(\zeta)\cos\left(\zeta\varkappa - g^{\frac{1}{2}}\zeta^{\frac{1}{2}}t\right)\right]d\zeta$$

$$+\left[\int_0^{\infty} e^{\zeta y}\bar{P}_0(\zeta)\cos(\zeta\varkappa + g^{\frac{1}{2}}\zeta^{\frac{1}{2}}t)d\zeta\right] \;\ldots.(3.2.4)$$

The evaluation of definite integrals of this type is in general, troublesome and can often be achieved only by the use of numerical or approximate methods. We shall return to this problem in the next section. For the moment we shall confine our attention to the evaluation of these integrals in a special case-that in which the waves are generated by the application of an impulsive pressure to a single point, the origin, say.

In this case we may take $P_0\ (\varkappa) = P\,\partial\ (\varkappa)$, where P is a constant , so that $\bar{P}_0\ (\zeta) = (2\pi)^{-\frac{1}{2}}$ P . when y = o , it follows from equation (3-2-4) that

$$\emptyset(x, o, t) = \frac{P}{2\pi\rho}\int_0^\infty \left[\cos\left(\zeta\varkappa - g^{\frac{1}{2}}\zeta^{\frac{1}{2}}t\right) + \cos\left(\zeta\varkappa + g^{\frac{1}{2}}\zeta^{\frac{1}{2}}t\right)\right] d\zeta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2.5)$$

Substituting (η^2) for $g\zeta$ in these integrals, we find that.

$$\emptyset(\varkappa, O, t) = -\frac{P}{\pi\,\rho g}\ \frac{dJ}{dt} \ \ldots\ldots (3.2.6)$$

where (J) denotes the integral

$$J = \int_0^\infty \left[\sin\left(\frac{\eta^2\varkappa}{g} + \eta t\right) - \sin\left(\frac{\eta^2\varkappa}{g} - \eta t\right)\right] d\eta \ldots . (3.2.7)$$

Making the substitutions

$$\zeta = \frac{\varkappa^{\frac{1}{2}}}{g^{\frac{1}{2}}}\left(\eta \mp \frac{gt}{2\varkappa}\right), \omega = \left(\frac{gt^2}{4\varkappa}\right)^{\frac{1}{2}}$$

we find that

$$J = -\frac{2g^{\frac{1}{2}}}{x^{\frac{1}{2}}}\int_0^\omega \sin(\ \omega^2 - \zeta^2\ ) d\zeta \ldots . (3.2.8)$$

From which it follows immediately that

$$\frac{dJ}{dt} = -\left(\frac{g}{\varkappa}\right)^{\frac{3}{2}} t \int_0^{\omega} \cos(\ \omega^2 - \zeta^2\ )d\zeta \ \rightarrow (3.2.9)$$

Substituting from equation (3.2.9) into equation (3.2.6) , we obtain for the velocity potential

$$\emptyset(\varkappa, o, t) \ = \frac{P}{\pi\ \rho} \frac{g^{\frac{1}{2}} t}{\varkappa^{\frac{3}{2}}} \ \int_0^{\omega} \ \cos(\ \omega^2 - \ \zeta^2\ )d\zeta \ \ldots\ldots (3.2.10)$$

With the usual notation for Fresnel's integrals,

$$C(u) = \int_0^{u} \cos(\ \frac{1}{2}\ \pi\ \varkappa^2\ )\ d\varkappa$$

$$S(u) = \int_0^{u} \sin(\ \frac{1}{2}\ \pi\ \varkappa^2\ )\ d\varkappa \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2.11)$$

we may write the solution (3-2-9) in the form

$$\emptyset(\varkappa, o, t) \ = \frac{Pu}{\rho \varkappa} \left[\cos(\frac{1}{2}\ \pi\ u^2\ )\ C(u) + \sin\left(\frac{1}{2}\ \pi\ u^2\ \right) S(u)\right] \rightarrow (3.2.12)$$

where

$$u^2 \ = \ \frac{g t^2}{2\pi\ \varkappa} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2.12)$$

The value of the velocity potential along y = o can therefore be determined readily from the calculated values of the Fresnel integrals C(u) and S(u) . The variation of these functions with (u) is shown in Fig. 1 and Fig 2 in page (54) . A simple geometrical interpretation of this result is, however, possible.

Equation (3.2.12) shows that we may consider the velocity potential (∅) to be the real part of the expression

$$\frac{Pu}{p\varkappa}\ e^{-\frac{1}{2}\pi i u^2} [C(u) + \ i\ S(u)]$$

If we now plot the curve whose freedom equations are

$$\varkappa = C(u) , y = S(u)$$

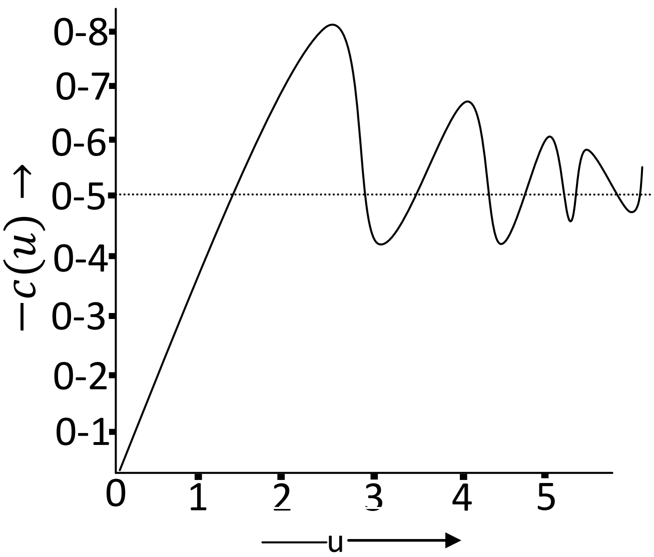
we get the well – known cornu spiral (Fig 3) so that, if (P) is the point on the spiral corresponding to the value u, we may write

$$C(u) + iS(u) = re^{i\theta}$$

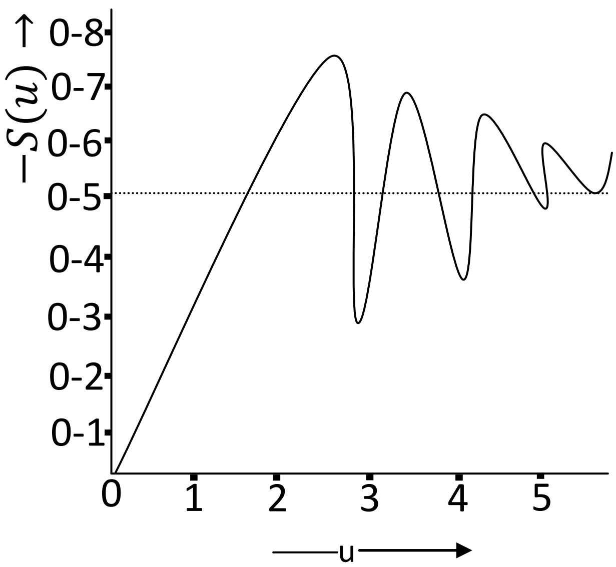
where (r) denotes the distance (OP) of P from the origin and (θ) is the angle ($\varkappa$ O P) , as shown in Fig 3 in page (55) With this notation (Ø) is the real part of the expression [1]

$$\frac{Pur}{\rho x}\ e^{-i(\frac{1}{2}\pi u^2-\theta\ )}$$

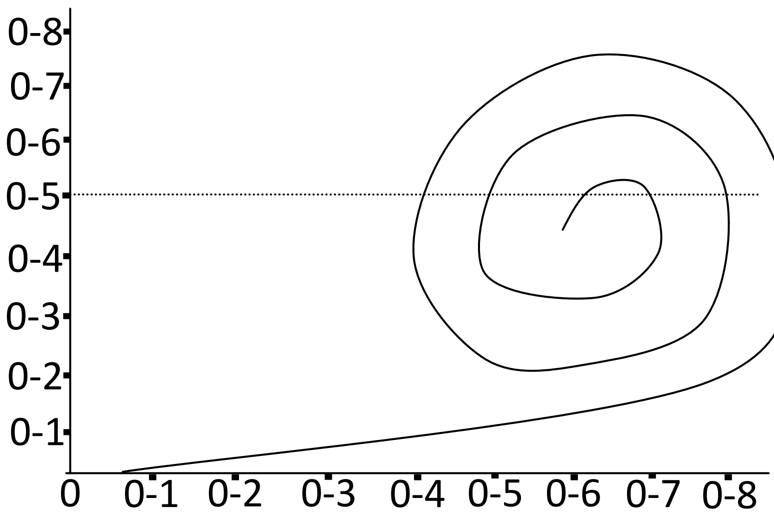
The variation with (u) of Fresnel's integrals C(u) and S(u)



(Fig 1)



(Fig2)



The Cornu spiral with freedom equations

$\mathcal{X}$ = C(u), y = S(u)

(Fig3)

from which it follows at once that

$$\emptyset \ = \frac{Pur}{\rho x} \cos(\frac{1}{2}\ \pi\ u^2 - \theta\ )$$

Now, since C (∞) =s(∞)= $\frac{1}{2}$, we see that for large values of (u) the point (P) approaches the limit point $\acute{0}$ ($\frac{1}{2}$, $\frac{1}{2}$) of the spiral and (r = $0\acute{0}$ = $\frac{1}{\sqrt{2}}$, θ = $\frac{\pi}{4}$) so that for large values of $gt^2/\varkappa$ we have the asymptotic expression

$$\emptyset(\varkappa, O, t) \sim \frac{P}{2\ \rho}\left(\frac{gt^2}{\pi n^3}\right)^{\frac{1}{2}} \cos\left(\frac{gt^2}{4\varkappa} - \frac{\pi}{4}\right) \ldots\ldots (3.2.14)$$

for the velocity potential on y = o.

Using the relations (3-1-5) and (3-2-13) we obtain $\eta = \ (2\pi g\varkappa)^{-\frac{1}{2}} \frac{\partial\emptyset}{\partial u}$

so that from the solution (3-2-12) we obtain the equation

$$\eta = \frac{P}{\rho\left(2\pi g \varkappa^{\frac{3}{2}}\right)^{1/2}}\left[C(u)\left[\cos(\frac{1}{2}\pi u^2) - \pi u^2 \sin(\frac{1}{2}\pi u^2)\right]\right]$$

$$+ \left[S(u)\left[\sin\left(\frac{1}{2}\pi u^2\right) + \pi u^2 \cos(\frac{1}{2}\pi u^2)\right]\right] + 1 \ldots (3.2.15)$$

for the elevation of the free surface.for large values of (u) we may take C(u) = S(u) = $\frac{1}{2}$ and neglect terms of order (u) and (1) in comparison with those of orderU^2. Equation (3.2.15) then reduces to

$$\eta \sim \frac{P g^{\frac{1}{2}} t^2}{4\pi^{\frac{1}{2}} \rho \varkappa^{\frac{5}{2}}} \cos\left(\frac{g t^2}{4\varkappa} + \frac{\pi}{4}\right) \ldots \ldots \ldots (3.2.16)$$

for large values of $gt^2/\varkappa$.[1]

(3-3) wave –Propagation in Two Dimensions

We may next consider some cases of wave. Propagation in two horizontal dimensions x,y .

The axis of Z being drawn vertically upwards, we have, on the hypothesis of infinitely small motion.

$$\frac{\mathrm{P}}{\rho} = \frac{\partial \emptyset}{\partial \mathrm{t}} - \mathrm{gZ} + \mathrm{F(t)} \ldots \ldots \ldots (3.3.1)$$

Where (Ø) satisfies

$$(\nabla^2 \emptyset) = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.3.2)$$

The arbitrary function F(t) may be supposed merged in the value of $(\partial \emptyset / \partial t)$. [6]

If the origin be taken in the undisturbed surface, and if (ξ) denoet the elevation at time (t) above this level, the pressure – condition to be satisfied at the surface is

$$\xi = \frac{1}{g}\left[\frac{\partial\phi}{\partial t}\right]_{Z=0} \dots\dots\dots..(3.3.3)$$

and the kinematical surface – condition is

$$\frac{\partial\xi}{\partial t} = -\left[\frac{\partial\phi}{\partial z}\right]_{Z=0} \dots\dots\dots..(3.3.4)$$

Hence, for Z = o , we must have

$$\frac{\partial^2\phi}{\partial t^2} + g\,\frac{\partial\phi}{\partial Z} = 0 \dots\dots..(3.3.5)$$

or, in the case of simple-harmonic motion ,

$$\sigma^2\phi = g\,\frac{\partial\phi}{\partial Z} \dots\dots\dots(3.3.6)$$

If the time – factor be ($e^{i(\sigma t+\varepsilon)}$) . [6]

The fluid being supposed to extend to infinity horizontally and down-wards, we may briefly examine, in the first place, the effect of a local initial disturbance of the surface , in the case of symmetry about the origin. [6]

The typical solution for the case of initial rest is easily seen, on reference to be

$$\left.\begin{aligned} \phi &= g\,\frac{\sin\sigma t}{\sigma}\,e^{kZ}\,J_0\,(k\varpi) \\ \xi &= \cos\sigma t\,J_0\,(k\varpi) \end{aligned}\right\} \dots(3.3.7)$$

Provided

$$\sigma^2 = gk \dots\dots\dots.(3.3.8)$$

To generalize this, subject to the condition of symmetry, we have recourse to the theorem $f(\varpi) = \int_0^\infty J_0(k\varpi)kdk \quad \int_0^\infty f(\propto) J_0(k \propto) \propto d \propto$ $\cdots \ldots \ldots . .(3.3.9)$

Thus corresponding to the initial conditions,

$$\xi = f(\varpi), \emptyset_0 = 0 \ldots \ldots \ldots (3.3.10)$$

We have

$$\emptyset = g \int_0^\infty \frac{\sin \sigma t}{\sigma} \mathrm{e}^{\mathrm{kZ}} J_0(k\varpi)\mathrm{kdk} \int_0^\infty f(\propto) J_0(k \propto) \propto \mathrm{d} \propto$$

and

$$\xi = \int_0^\infty \cos \sigma t \ J_0(k\varpi)\mathrm{kdk} \int_0^\infty f(\propto) J_0(k \propto) \propto \mathrm{d}\,\alpha \ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.3.11)$$

If the initial elevation be concentrated in the immediate neighborhood of the origin, then, assuming. [6]

$$\int_0^\infty f(\propto) 2\pi \propto \mathrm{d} \propto = 1 \ldots \ldots . .(3.3.12)$$

We have

$$\emptyset = \frac{g}{2\pi} \int_0^\infty \frac{\sin \sigma t}{\sigma} \mathrm{e}^{\mathrm{kZ}} J_0(k\varpi)\mathrm{kdk} \ldots (3.3.13)$$

Expanding, and making use of (8) , we get

$$\emptyset = \frac{gt}{2\pi} \int_0^\infty \left\{ k - \frac{gt^2}{3!} k^2 + \frac{(gt^2)^2}{5!} k^3 - \cdots \right\} \mathrm{e}^{\mathrm{kZ}} J_0(k\varpi)\mathrm{dk} \ \ldots \ldots \ldots \ldots (3.3.14)$$

If we put

$$Z = \text{-r} \cos\theta \,,\, \varpi \;=\; \text{r} \sin\theta \,,\, \dots \dots .. (3.3.15)$$

we have

$$\int_0^{\infty} \mathrm{e}^{\mathrm{kZ}}\, J_0\,(k\varpi)\mathrm{dk} \;= \frac{1}{r}\,,\dots..(3.3.16)$$

and thence

$$\int_0^{\infty} \mathrm{e}^{\mathrm{kZ}}\, J_0\,(k\varpi)\mathrm{k}^{\mathrm{n}}\, dk = \left(\frac{\partial}{\partial Z}\right)^n \frac{1}{r} \;=\; n!\,\frac{P_n(\mu)}{\mathrm{r}^{\mathrm{nt1}}} \dots (3.3.17)$$

Where $\mu = \cos\theta\;.Hence$

$$\emptyset \;=\; \frac{gt}{2\pi}\left\{\frac{P_1(\mu)}{\mathrm{r}^2} - \frac{gt^2}{3!}\;\frac{2!\,P_2(\mu)}{\mathrm{r}^3} + \frac{(gt^2\,)^2}{5!}\;\frac{3!\,P_3(\mu)}{\mathrm{r}^4}\right\} \dots \dots ... (3.3.18)$$

Form this the value of (ξ) is to be obtained by (3.3.3). It appears from that.

$$\mathrm{P}_{2n+1}\,(0) = 0\,,\,\mathrm{P}_{2n}\,(0) = (\text{-}1)^{\mathrm{n}}\;\frac{1.3\dots(2\mathrm{n}-1)}{2.4\dots2\mathrm{n}}\;,\dots(3.3.19)$$

whence :

$$\xi = \frac{1}{2\pi\varpi^2}\left\{\begin{array}{c}\frac{1^2}{2!}\;\frac{gt^2}{\varpi}\;-\;\frac{1^2.\,3^2}{6!}\left(\frac{gt^2}{\varpi}\right)^3 + \\ \frac{1^2.\,3^2.\,5^2}{10!}\left(\frac{gt^2}{\varpi}\right)^5 - \end{array}\right\} \dots \dots \dots .(3.3.20)$$

It follows that any particular phase of the motion is associated with a particular value of (gt^2/ϖ) , and thence that the various phases travel radially outwards from the origin, each with a constant acceleration. [6]

No exact equivalent for (3.3.20) , analogous to the formula which was obtained in the two-dimensional from of the problem, and accordingly suitable for discussion in the case where(gt^2/ϖ)is large, has been discovered. An approximate value may however be obtained by kelvin's method. Since J_0 (Z) is a fluctuating function which tends as (Z) increases to have the same period (2π) as sin (Z), the elements of the integral in (3-3-13) will for the most part cancel one another with the exception of those for which

$$\mathrm{t\,d}\,\sigma\,/\,\mathrm{dk} = \varpi \;\text{, or } \mathrm{k}\,\varpi = gt^2/4\varpi \;\;\ldots.(3.3.21)$$

nearly. Now when (k ϖ)   is large we have

$$J_0\,(\mathrm{k}\,\varpi) = (\frac{2}{\pi k\varpi})^{\frac{1}{2}}\sin\,(kw + \frac{1}{4}\,\pi\,)\ldots.(3.3.22)$$

Approximately and we may therefore replace (3.3.13) by

$$\emptyset\; = \frac{g^{\frac{1}{2}}}{2^{\frac{3}{2}}\pi^{\frac{3}{2}}\varpi^{\frac{1}{2}}}\int_0^{\infty} \mathrm{e}^{\mathrm{kZ}}\;\;\cos\Big(\sigma\,t - k\varpi - \frac{1}{4}\,\pi\;\Big)d\,k\;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.3.23)$$

When putting now Z = o , we find as the surface value of (∅)

$$\emptyset_0\; = \frac{g^{\frac{1}{2}}}{2\pi\varpi^{\frac{1}{2}}\sqrt{/td^2\sigma\,/\,dk^2/}}\sin(\sigma t - k\varpi\,),\ldots..(3.3.24)$$

where (k) and (σ) are to be expressed in terms of (ϖ) and (t) by means of (3.3.8) and (3.3.21). Not has here been taken of the fact that ($d^2\sigma/dk^2$) is negative. Since

$$\sigma t = (gkt^2)^{\frac{1}{2}} = 2kw,$$

$$\frac{td^2\sigma}{dk^2} = -\frac{1}{4}g^{\frac{1}{2}}tk^{-\frac{3}{2}} = -2\varpi^3 /gt^2 \text{ ,.... } (3.3.25)$$

we have

$$\emptyset_0 = \frac{gt}{2^{\frac{3}{2}}\pi\varpi^2}\sin\frac{gt^2}{4\varpi} \text{ ... ... } (3.3.26)$$

The surface elevation is then given by (3.3.3). Keeping, for consistency, only the most important term, we find

$$\xi = \frac{gt^2}{2^{\frac{5}{2}}\pi\varpi^3}\cos\frac{gt^2}{4\varpi} \rightarrow (3.3.27)$$

Which agrees with the result obtained, in other ways, by Cauchy and Poisson.[6]

It is not necessary to dwell on the interpretation, which will be readily understood from what has been said with respect to the two- dimensional case. The consequences were worked out in some detail by Poisson on the hypothesis of an initial paraboloidal depression.

When the initial data are of impulse, the typical solution is

$$\left.\begin{aligned} \rho\emptyset &= \cos\ \sigma t\ \ \mathrm{e}^{\mathrm{kZ}}\ J_0\ (k\varpi) \\ \xi &= -\frac{\sigma}{\mathrm{g}\rho}\ \sin\ \sigma t\ J_0\ (k\varpi) \end{aligned}\right\} \text{..... } (3.3.28)$$

Which being generalized, gives , for the initial conditions

$$\rho\emptyset_0 = F(\varpi), \quad \xi = 0 \text{ ... ... } (3.3.29)$$

the solution

$$\emptyset = \frac{1}{\rho}\int_0^\infty \cos\sigma t\, \mathrm{e}^{\mathrm{kZ}}\ J_0\ (k\varpi)\mathrm{k}\, dk \int_0^\infty F(\propto)\ J_0\ (k \propto) \propto \mathrm{d} \propto$$

and

$$\xi = -\frac{1}{g\rho}\int_0^{\infty} \sigma \sin \sigma t \;\; J_0\,(k\varpi)\mathrm{k}\, dk \int_0^{\infty} F(\propto)\;J_0\,(k \propto) \propto \mathrm{d} \propto \;\dots\dots\dots\dots(3.3.30)$$

In particular, for a concentrated impulse at the origin, such that

$$\int_0^{\infty} F(\propto)\, 2\pi \;\propto \mathrm{d} \propto = \;1 \dots\dots\dots..(3.3.31)$$

we find

$$\emptyset \;= \frac{1}{2\pi\rho}\int_0^{\infty} \cos \sigma t\, \mathrm{e}^{\mathrm{kZ}}\; J_0\,(k\varpi)\mathrm{k}dk \;\;\dots..(3.3.32)$$

Since this may be written

$$\emptyset \;= \frac{1}{2\pi\rho}\;\frac{\partial}{\partial \mathrm{t}}\int_0^{\infty} \frac{sin\sigma t}{\sigma}\;\;\mathrm{e}^{\mathrm{kZ}}\; J_0\,(k\varpi)\mathrm{k}dk \;\;\dots.(3.3.33)$$

We find, performing $(\frac{1}{g\rho}\;.\frac{\sigma}{\sigma t})$ on the results contained in (3.3.18) and (3.3.20),

$$\left.\begin{aligned}\emptyset \;&= \;\frac{1}{2\pi\rho}\left\{\frac{P_1(\mu)}{r^2} - \;\frac{gt^2}{2!}\;\;\frac{2!\,P_2(\mu)}{r^3} + \;\frac{(gt^2)^2}{4!}\;\;\frac{3!\,P_3(\mu)}{r^4} - \cdots\right\}\\ \xi &= \frac{t}{2\pi\rho\varpi^3}\left\{1 - \frac{1^2.3^2}{5!}\;\;(\frac{gt^2}{\varpi})^2 \;+\; \frac{1^2.3^2.5^2}{7!}\left(\frac{gt^2}{\varpi}\right)^4 - \right\}\end{aligned}\right\}$$

……………………………………….(3.3.34)

Again, when $(\frac{1}{2}\;gt^2/\varpi)$ is large, we have, in place of (3.3.27)

$$\xi = -\frac{gt^3}{2^{\frac{7}{2}}\pi\rho\varpi^4}\;\sin\frac{gt^2}{4\varpi} \dots\dots\dots(3.3.35)$$

(3-4) Slow motion of a viscous Fluid

We shall now turn to the discussion of problems in which the viscous nature of the fluid plays an important role. The complete equations of motion of an incompressible viscous fluid [equations $(\frac{\partial v}{\partial t} - 2vxw + grad\ (\Omega + \frac{1}{2}\ r^2 + \frac{P}{\rho})$] are too difficult to solve exactly in any particular problem because they are nonlinear.

In the case of very slow motions the equations reduce, in first approximation, to linear equations since the terms v x w and V^2 will be of the second order. In the sub sequent pages we shall discuss certain boundary value problems which may be treated under this approximation and then consider how far the methods of Fourier analysis may be applied to the non linear case.

(3-5) Diffusion of vorticity

In the two – dimensional case in which either the motion is very slow or (ξ) is a function of (φ) we may write equation :

$$\frac{\partial \xi}{\partial \mathrm{t}} + u\ \frac{\partial \xi}{\partial \varkappa} + \mathrm{v}\ \frac{\partial \xi}{\partial \mathrm{y}} = v\nabla_1^2 \xi \ \ in\ the\ form$$

In the form

$$\frac{\partial \xi}{\partial \mathrm{t}} = v\nabla_1^2 \xi \quad \dots\dots\dots\dots\dots\dots\dots..(3.5.1)$$

Which is the equation for the two-dimensional flow of heat.

If the fluid fills the whole space and if, when t = o.

$$\xi = \xi_0\,(\varkappa\,, y) \ldots\ldots\ldots\ldots..\ldots (3.5.2)$$

Then we may solve equation (3.5.2) by multiplying both sides by $(\frac{1}{2\pi})\ \mathrm{e}^{\mathrm{i}(\mathrm{w}_1\varkappa+\mathrm{w}_2\mathrm{y})}$ and in tegrating throughout the whole any plane. In this way we obtain the equation :

$$\frac{dZ}{d\mathrm{t}} = -v\,(\mathrm{w}_1^2 + \mathrm{w}_2^2) Z\left(\mathrm{w}_1\,, \mathrm{w}_{2\,,}\,\mathrm{t}\right) \ldots\ldots (3.5.3)$$

for the rate of change of the Fourier transform.

$$Z\left(\mathrm{w}_1\,, \mathrm{w}_{2\,,}\,\mathrm{t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi(\varkappa\,, y\,, t)\ \mathrm{e}^{\mathrm{i}(\mathrm{w}_1\varkappa+\mathrm{w}_2\mathrm{y})}\ \mathrm{d}\varkappa\ \mathrm{dy}$$

of the vorticity $\xi(\varkappa\,, y\,, t)$. The solution of equation (3.5.3) satisfying the initial condition (3.5.2) is

$$\mathrm{Z} = \mathrm{Z}_0\ \mathrm{e}^{-\mathrm{v}\left(\mathrm{w}_1^2+\mathrm{w}_2^2\right)}\ t$$

Where Z_o is the Fourier transform of the function $\xi_0\,(\varkappa\,, y)$. Hence, making use of the inversion theorem for double Fourier transforms [$n = 2$ in equation (3.5.2) , we have

$$\xi(\varkappa\,, y\,, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Z}_0\,(\mathrm{w}_1\,, \mathrm{w}_2\ )\ \mathrm{e}^{-\mathrm{v}\left(\mathrm{w}_1^2+\mathrm{w}_2^2\right)}\ t \\ -\ \mathrm{i}(\mathrm{w}_1\varkappa + \mathrm{w}_2\mathrm{y}) dw_1 dw_2$$

In the special case in which we may write

$$\xi_0\,(\varkappa\,, y) = f(\varkappa) g(y)$$

this solution reduces to the form

$\xi(\varkappa , y , t)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{F(w_1\,)\,e^{-vw_1^2t-iw_1\varkappa}}\, dw_1 \int_{-\infty}^{\infty} \mathrm{G(w_2\,)\,e^{-vw_2t-iw_1 y}}\, dw_2$$

in which F and G ar the usual (one-dimensional) transforms of the function $f(\varkappa)$ and $g(y)$. For example, if initially there is avortex sheet in the plane $\varkappa$ =o , we may write $\xi_0 = U\delta\,(\varkappa)$ so that , taking $f(\varkappa)$ = $U\delta\,(\varkappa)$ and g(y) = 1 , we readily calculate that $F(w_1) = (2\pi)^{-\frac{1}{2}} U$ and $G(w_2) = (2\pi)^{\frac{1}{2}}\ \delta\,(w_2)$

giving

$$\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e^{-vw_1^2t-iw_1\varkappa}}\, dw_1 \ = \frac{u}{2\sqrt{\pi v t}}\, e^{-\varkappa^2}/4vt$$

From the relation $\frac{\partial u}{\partial \varkappa} = 2\xi, we\ find\ that$

$$u(\varkappa\ \ , t) = \frac{2u}{\sqrt{\pi}} \int_{0}^{\varkappa/2(vt)^{\frac{1}{2}}} e^{-v^2}\, dv$$

If, on the other hand, there wase initially avortex filament of strength (k) along the [1]

$$Z - axis\ , then\ \xi_0\ = \frac{1}{2}\ k\ \delta\ (\varkappa)\delta\ (y) and\ so$$

$Z_0 = \ k/4\pi\ \ , whence$

$$\xi = \frac{k}{8\pi^2}\ \int_{-\infty}^{\infty} \mathrm{e^{-vw_1^2t-iw_1\varkappa}}\, dw_1\ \int_{-\infty}^{\infty} \mathrm{e^{-vw_2^2t-iw_2 y}}\, dw_2$$

The integrations are elementary and give

$$\xi = \frac{k}{8\pi v t}\ e^{-(\varkappa^2+y^2)/4vt}$$

In the case of axial symmetry about the (Z) axis, equation (3.5.1) may be written as :

$$\frac{\partial \xi}{\partial \mathrm{t}} = v(\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial \mathrm{r}})$$

Multiplying both sides of this equation by $rJ_0(wr)$ and integrating with respect to (r) from (o) to (∞) , we find that the trans form :

$$\bar{\xi}\,(w,t) = \int_0^\infty rJ_0\,(wr)\xi\,(r,t)dr.$$

Satisfies the first – order equation

$$\frac{d\bar{\xi}}{dt} = vw^{2\bar{\xi}}$$

Subject to the initial condition

$\xi\ = \mathrm{f(r)}$, this equation has the solution

$$\xi\ = \bar{f}(w)\ e^{-vw^2t}$$

So that, by the application of the Hankel inversion we find

$$\xi\ = \int_0^\infty w, J_0\,(wr)\ e^{-vw^2t}\ \bar{f}(w)dw$$

For the case of aline filament we have

$$f(r) = k\delta(r)/4\pi r, so\ that\ \bar{f}(w) = k/4\pi\ and$$

$$\xi = \frac{k}{4\pi}\int_0^\infty w, J_0\,(wr)\ e^{-vw^2t}\ f(w)dw$$

Making use of equation :

$$\frac{\partial}{\partial \mathrm{t}} \nabla_1^2\Psi + \left(\frac{\partial\Psi}{\partial \mathrm{y}} \cdot \frac{\partial\Psi}{\partial \varkappa} \frac{\partial}{\partial y}\right)\nabla_1^2\Psi = \nu\nabla_1^4\Psi$$

Appendix A , with V = o , we get the same result as we did by employing the theory of two-dimensional Fourier transform.[1]

Chapter Four

Chapter Four

Fluid Motion

(4-1) Motion of a Viscous Fluid Contained between Two Infinite Coaxal Cylinders

As an example of the use of finite Hankel transforms in the discussion of boundary value problems in the theory of the motion of viscous fluids we consider the motion of such a fluid contained between two infinite coaxal cylinders. We regard the cylinders as being of infinite length and, as is usually done in problems of this type, assume that the velocities involved are sufficiently small for their squares to be neglected. [1]

If we take the (Z) axis along the common axis of the cylinders and the ($\varkappa$) and (y) axes normal to it, then , denoting the components of the velocity in the ($\varkappa$) and (y) directions by (u) and (v) , respectively, and neglecting terms involving the squares of the components of velocity, we may write the basic equation:

$$\frac{\partial \mathrm{v}}{\partial \mathrm{t}} - 2v \times w + grad\left(\Omega + \frac{1}{2}v^2 + \frac{\mathrm{P}}{\rho}\right) - v\nabla^2 v = 0$$

In the form

$$\frac{\partial P}{\partial \varkappa} = \mu\left(\frac{\partial^2 \mathrm{u}}{\partial \varkappa^2} + \frac{\partial^2 \mathrm{u}}{\partial y^2}\right) - \rho\frac{\partial \mathrm{u}}{\partial t}$$

$$\frac{\partial P}{\partial y} = \mu\left(\frac{\partial^2 \mathrm{v}}{\partial \varkappa^2} + \frac{\partial^2 \mathrm{v}}{\partial y^2}\right) - \rho\frac{\partial \mathrm{v}}{\partial t}$$

Where (p) is the pressure at the point ($\varkappa, y$) in the fluid and (μ) and (ρ) denote respectively, the coefficient of viscosity and density of the fluid, for rotational motion we may write

$$U = -V \sin \theta, \;\; v = v \cos \theta$$

so that , making use of the relations

$$\frac{\partial}{\partial \varkappa} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}$$

We may put the equations of motion in the form

$$\frac{\partial P}{\partial r} - \frac{1}{r} \frac{\partial P}{\partial \theta} \tan\theta = -\tan\theta \left[\mu \left(\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) - \rho \frac{\partial \mathrm{v}}{\partial t} \right]$$

$$\frac{\partial P}{\partial r} - \frac{1}{r} \frac{\partial P}{\partial \theta} \cot\theta = \cot\theta \left[\mu \left(\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) - \rho \frac{\partial \mathrm{v}}{\partial t} \right]$$

Further , if the motion is symmetrical about the (Z) axis,

$$\frac{\partial P}{\partial \theta} = 0 \dots \dots \dots \dots \dots \dots \dots \dots \dots (4.1.1)$$

from which it follows immediately that the above equations are equivalent to the pair

$$\frac{\partial P}{\partial r} = 0 \dots \dots \dots \dots \dots \dots \dots \dots .. (4.1.2)$$

and

$$\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} = \frac{1}{v}\frac{\partial v}{\partial t}\ , a \leq r \leq b, t > 0\ \ldots\ldots (4.1.3)$$

where (v) denotes the kinematic viscosity (μ/ρ). the inequalities in equation (4.1.3) merely express the fact that we are interested only in the region bounded by cylinders of radiis (a) and (b) and in positive values of the time. It is an immediate consequence of equations (4.1.1) and (4.1.2) that , with the assumptions we have made , the pressure (p) is a constant throughout the fluid. This is hardly surprising when we remember that the squares of the velocities have been neglected. The fundamental equation $(4.1.3)$ may be derived directly as follows . Consider an annular element of fluid of radius (r) and thickness (Δr), and let (ω) be the angular velocity of the fluid. Then the frictional force per unit area on a cylindrical shell of radius (r), is $2\pi r r\mu(\partial\omega/\partial r)$ per unit length, and the equation of motion of the annular element is

$$2\pi r^3\ \rho\Delta r \frac{\partial \omega}{\partial t} = \frac{\partial}{\partial r}\left(2\pi r^3 \mu\ \frac{\partial \omega}{\partial r}\right)\Delta \mathrm{r}$$

$$or\ \frac{\partial^2 \omega}{\partial r^2} + \frac{3}{r}\frac{\partial \omega}{\partial r} = \frac{1}{v}\frac{\partial \omega}{\partial t}$$

On putting($\omega = V/r$) we immediately obtain equation ... $(4.1.3)$. [1]

(4.2) Motion when the outer cylinder rotates at a constant speed

Suppose viscous fluid is contained between two infinite coaxal cylinders of radii (a) and (b) and that the fluid is set in motion by the outer cylinder (r = b) starting to rotate with uniform angular velocity (Ω) at the instant (t = 0) , the inner cylinder being kept at rest. Then (v = Ωb) when (r = b) , and (v = 0) when (r = a). Multiplying the left hand side of equation

$$\frac{\partial^2 \mathrm{v}}{\partial \mathrm{r}^2}+\frac{1}{\mathrm{r}}\frac{\partial \mathrm{v}}{\partial \mathrm{r}}=\frac{\mathrm{v}}{\mathrm{r}^2}=\frac{1}{\mathrm{v}}\frac{\partial \mathrm{v}}{\partial \mathrm{t}}, \mathrm{a} \le \mathrm{r} \le \mathrm{b}, \mathrm{t} > 0 \quad \ldots \ldots (4.2.1).$$

by the function :

$$\mathrm{r}\,[\,\mathrm{J}_1(\xi_i r) G_1(\xi_i a) - G_1(\xi_i r)\mathrm{J}_1(\xi_i a)]$$

Where (ξ_i) a positive root of the equation

$$\mathrm{J}_1(\xi_i b) G_1(\xi_i a) - G_1(\xi_i b)\mathrm{J}_1(\xi_i a) = 0 \quad \ldots \ldots (4.2.2)$$

then , using , equation

$$\mathrm{H}_\mu\left(\frac{d^2 f}{d\varkappa^2}+\frac{1}{\varkappa}\frac{df}{d\varkappa}-\frac{\mu^2}{\varkappa^2}\right)$$
$$=\frac{J\mu(\xi_i a)}{J\mu(\xi_i b)} f(b) - f(a) - \xi_i^2 H\mu(f) \ldots (4.2.3)$$

We find that

$$\int_a^b \mathrm{r}\,[\,\mathrm{J}_1(\xi_i r) G_1(\xi_i a) - J_1(\xi_i a)\mathrm{G}_1(\xi_i r)]\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}-\frac{v}{r^2}\right)\mathrm{dr}$$
$$= -\xi_i^2 \bar{V}_H + V(a) - V(b)\frac{\mathrm{J}_1(\xi_i a)}{\mathrm{J}_1(\xi_i b)} \ldots \ldots \ldots \ldots \ldots (4.2.4)$$

Where $\bar{V}_{\mathrm{H}}$ denotes the finite Hankel transform

$$\overline{V}_H = \int_a^b r\,[\,J_1(\xi_i r)G_1(\xi_i a) - J_1(\xi_i a)G_1(\xi_i r)]V(r)dr \ldots (4.2.5)$$

of the velocity V(r). Substituting the boundary conditions V(a) = 0 , V(b) = (Ωb), we see that the right – hand side of equation (4.2.4) becomes simply

$$-\,\xi_i^2 V_H - \Omega b\,\frac{J_1(\xi_i a)}{J_1(\xi_i b)}$$

Thus we obtain the first – order ordinary linear equation :

$$\frac{1}{V}\,\frac{dV_H}{dt} + \ \xi_i^2 V_H + \ \Omega b\,\frac{J_1(\xi_i a)}{J_1(\xi_i b)} = 0 \ldots \ldots \ldots \ldots (4.2.6)$$

for the determination of the finite Hankel transform ($\bar{V}_H$). [1]

Since the outer cylinder starts from rest when t = 0 , we have the initial condition $\bar{V}_H = 0$ when t = 0 , so that the appropriate solution of equation (4.2.6) is

$$\overline{V}_H = \frac{\Omega b}{\xi_i^2}\,\frac{J_1(\xi_i a)}{J_1(\xi_i b)}\left(1 - e^{-v\xi_i^2 t}\right) \ldots \ldots \ldots (4.2.7)$$

Substituting this value for (V_H) into the inversion formula for the finite Hankel transform employed her we have.

$$V(r) = \ 2\Omega b \sum \frac{J_1(\xi_i a)J_1(\xi_i b)}{J_1^2(\xi_i a) - J_1^2(\xi_i b)}(1 - e^{-v\xi_i^2 t})$$

$$[\,J_1(\xi_i r)G_1(\xi_i a) - G_1(\xi_i r)J_1(\xi_i a)] \ldots\ldots (4.2.8)$$

where the sum is taken over all the positive roots of equation (4.2.2). [1]

Now from equations

$$f(\varkappa) = \sum \frac{2\xi_i^2 J_\mu^2(\xi_i b) f_H(\xi_i)}{J_\mu^2(a\xi_i) - J_\mu^2(b\xi_i)} \left[J_\mu(\varkappa\xi_i) G_\mu(a\xi_i) - J_\mu(a\xi_i) G_\mu(\varkappa\xi_i)\right] \ldots\ldots (4.2.9)$$

and

$$H_1 = \left(\frac{\varkappa^2 - a^2}{\varkappa}\right) = \frac{a^2 - b^2}{b\xi_i^2} \frac{J_1(\xi_i a)}{J_1(\xi_i b)} \ldots\ldots (4.2.10)$$

it follows at once that :

$$\frac{\Omega b^2}{r}\left(\frac{r^2 - a^2}{b^2 - a^2}\right) = -2\Omega b \sum_i \frac{J_1(\xi_i a) J_1(\xi_i b)}{J_1^2(\xi_i a) - J_1^2(\xi_i b)}$$

$$[J_1(\xi_i r) G_1(\xi_i a) - J_1(\xi_i a) G_1(\xi_i r)] \ldots\ldots (4.2.11)$$

so that we may write the solution finally in the form [1]

$$V(r) = \frac{\Omega b^2}{r}\left(\frac{r^2 - a^2}{b^2 - a^2}\right) + 2\Omega b \sum_i \frac{J_1(\xi_i a) J_1(\xi_i b)}{J_1^2(\xi_i a) - J_1^2(\xi_i b)} e^{-v\xi_i^2 t} \times \left[J_1(\xi_i r) G_1(\xi_i a) - G_1(\xi_i r) J_1(\xi_i r)\right] \ldots\ldots (4.2.12)$$

(4-3) Motion of a viscous Fluid under a surface load

In the discussion of the plastie reeoil of the earth after the disappearance of the Pleistocene ice sheets a boundary value problem in the theory of viscous fluids arises which can be solved by means of the theory of Hankel transforms. The curvature of the earth is neglected, and as a model we treat the motion of a semiinfinite, in compressible, viscous fluid under the action of aradially symmetrical pressure applied to the free surface. Since, in the case of the earth, we are dealing with extremely small accelerations and very high viscosity, we may neglect the inertial terms in the equations of motion in comparison with those arising from viscous forces.[1]

Neglecting the terms arising from the acceleration, the equations of motion of a fluid in a gravitational field may be written in the vector form.

$$\mu\nabla^2 \mathrm{V} = grad\ P - (0,0,pg) \ldots \ldots \ldots (4.3.1)$$

Provided that the positive (Z) axis is taken as pointing down ward. The velocity (v) at the point $(\varkappa, y, Z)$ in the fluid must also satisfy the equation of continuity. Transforming to cylindrical coordinates (r,Z, φ) and assuming cylindrical symmetry, we see that the equation (4.2.1) become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial vr}{\partial r}\right) - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial Z^2} = \frac{1}{\mu}\frac{\partial \bar{p}}{\partial r} \ldots \ldots (4.3.2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_Z}{\partial r}\right) + \frac{\partial^2 v_Z}{\partial Z^2} = \frac{1}{\mu}\frac{\partial \bar{p}}{\partial Z} \ldots \ldots (4.3.3)$$

where we have written ($\bar{P}$ = p - gρZ) similary the equation of continuity div (v = 0). transforms to

$$\frac{1}{r}\frac{\partial}{\partial r}(r\mathrm{V_r}) + \frac{\partial u_Z}{\partial Z} = 0 \ldots \ldots \ldots (4.3.4)$$

The components of stress associated with the (Z) direction are.

$$\sigma_Z = -P + 2\mu \frac{\partial v_Z}{\partial Z} \dots\dots\dots\dots\dots (4.3.5)$$

and

$$\tau_{rZ} = \mu \left(\frac{\partial v_r}{\partial Z} + \frac{\partial v_z}{\partial r}\right) \dots\dots.\dots (4.3.6)$$

The boundary conditions are that, on the free surface, the shearing stress (τ_{rZ}) is zero and that the normal component of stress (σ_Z) is equal to the applied pressure. In addition, it is assumed that at infinity the stresses and the components of velocity are zero. If we suppose that the equation of the free surface is Z = ξ(r,t) and that the equation of the undisturbed free surface is Z = 0, then we take (ξ) to be small in comparison with the other distances which enter into the problem.

Such as. for instance, the radius of the circle to which the load is applied. Just as. in the case of surface waves. we may take, at least in first approximation, the free surface to be Z = 0 . Thus we replace the value of ($^{\partial v_z}/_{\partial Z}$ ***at*** $Z = \xi$) by its value at Z=0, and similarly with the other quantities except ($g\rho Z = g\rho\xi$ (r,t)) . If we denote the applied pressure by σ(r,t) , we have $\sigma_{Z=} - \sigma(r,t)$ when Z = ξ (r,t). Or, by means of equation (4.3.5) with the " surface wave approximation".

$$\bar{P} + g\rho\xi(r,t) - 2\mu \frac{\partial v_Z}{\partial Z} - \sigma(r,t), \text{on } Z = 0 \dots\dots.(4.3.7)$$

The relation between (ξ) and (V_z) is that, at the free surface, the rate of change of (ξ) is equal to (V_z) ; hence we have

$$\frac{\partial \xi}{\partial t} = (V_z)_Z = 0 \dots\dots\dots\dots.(4.3.8)$$

Finally, the condition that the shearing stress on the free surface is Zero becomes, in this approximation. [1]

$$(\tau_{rZ})_{Z=0} = 0 \ldots\ldots\ldots.(4.3.9)$$

(4-4) Harmonic Analysis of Nonlinear Viscous Flow

In this part we have discussed problems in viscous fluid flow by assuming that the motion was so slow that the terms of the second order in the velocity could be neglected. Thus, in the discussion of the diffusion of vorticity we made use of the approximate equation

$$\frac{\partial \xi}{\partial t}\ V\nabla_1^2\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4.4.1)$$

instead of the exact equation

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial \varkappa}\ V\frac{\partial \xi}{\partial y} = V\nabla_1^2\xi \ldots\ldots..(4.4.2)$$

For the variation of the vorticity.In this section we shall consider this latter equation. Because of its nonlinearity this equation does not lend itself readily to the application of methods based on the theory of integral transforms, but some progress has recently been made in this direction by kampe de Feriet. It is possible to set up relations between the double Fourier transforms of the various physical quantities entering into the problem and then, with the help of these relations, to form and integrodifferential equation governing the behavior of the double transform of the vorticity. The study of this equation is very difficult but would seem to be the rational starting point for the rigorous discussion of the behavior of the transform of the vorticity and the related spectral function. Though the solution of particular boundary value problems by this method will be exceedingly complicated, it should be possible to investigate in this way the validity of Several of the working hypotheses which have been made in the past, such as the assumption that the big eddies have a tendency to degenerate into smaller ones. [1]

We shall first of all write down some important properties of double Fourier transforms of which we shall make use. If $f(\varkappa, y)$ is areal function of the variables ($\varkappa$) and (y), then its double Fourier transform

$$F(\omega_1, \omega_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\varkappa, y)\, e^{i(\omega_1 \varkappa + \omega_2 y)}\, dxdy \ldots\ldots (4.4.3)$$

has complex conjugate

$$F^*(\omega_1, \omega_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\varkappa, y)\, e^{-i(\omega_1 \varkappa + \omega_2 y)}\, dxdy \ldots.$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (4.4.4)$$

then, by a method similar to that employed in establishing the Faltung theorem we may readily show that:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\varkappa, y) g(\varkappa, y) e^{i(\theta, \varkappa + \theta_2 y)}\, dxdy =$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega_1, \omega_2) G^*(\omega_1 - \theta_1, \omega_2 - \theta_2) d\omega_1 d\omega_2 \ldots (4.4.5)$$

Where $f(\varkappa, y)$ and $g(\varkappa, y)$ are real function of ($\varkappa$) and (y). In particular, we shall make use of the special case g = f,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^2(\varkappa, y)d$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(\omega_1, \omega_2)|^2 d\omega_1 d\omega_2 \ldots\ldots\ldots.(4.4.6)$$

If we introduce the double Fourier transform $\Psi(\omega_1, \omega_{2,} t)$ of the stream function $\Psi(\varkappa, y, t)$ through the eguation

$$\Psi(\omega_1, \omega_{2\,,} t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\varkappa, y, t)\, e^{i(\omega_1 \varkappa + \omega_2 y)} \;\; dxdy$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots.(4.4.7)$$

Then it follows from equation

$$u = -\frac{\partial \Psi}{\partial y}\,, V = \frac{\partial \Psi}{\partial \varkappa}$$

that the Fourier transforms $\mathrm{U}(\omega_1, \omega_{2\,,}\mathrm{t})$ and $\mathrm{V}(\omega_1, \omega_{2\,,}\mathrm{t})$ of the velocity components $u(\varkappa, y, t)$ and $\mathrm{v}((\varkappa, y, t)$ satisfy the relations [1]

$$\mathrm{U}(\omega_1, \omega_{2\,,}\mathrm{t}) = i\omega_2\ \Psi(\omega_1, \omega_{2\,,}\mathrm{t})\mathrm{V}(\omega_1, \omega_{2\,,}\mathrm{t})$$

$$= -i\omega_1\, \Psi(\omega_1, \omega_{2\,,}\mathrm{t})$$

Similarly, by multiplying both sides of the equation

$$\xi = -\frac{1}{2}\,\nabla_1^2 \Psi$$

by $e^{i(\omega_1 \varkappa + \omega_2 y)}$and integrating over the whole $(\varkappa\ y)$ plane, we obtain the equation

$$Z = \frac{1}{2} (\omega_1^2, \omega_2^2)\Psi$$

for the double transform $Z(\omega_1, \omega_{2,}t)$ of the vorticity $\xi(\varkappa, y, t)$.

From these relations it follows immediately that

$$\left.\begin{aligned} \Psi(\omega_1, \omega_{2,}t) &= \frac{2}{\omega_1^2 + \omega_2^2} Z(\omega_1, \omega_{2,}t) \\ \mathrm{U}(\omega_1, \omega_{2,}t) &= \frac{2\mathrm{i}\,\omega_2}{\omega_1^2 + \omega_2^2} \mathrm{Z}(\omega_1, \omega_{2,}t) \\ \mathrm{V}(\omega_1, \omega_{2,}t) &= -\frac{2\mathrm{i}\,\omega_1}{\omega_1^2 + \omega_2^2} \mathrm{Z}(\omega_1, \omega_{2,}t) \end{aligned}\right\} \ldots.(4.4.8)$$

Now, from equation (4.4.6), we have that

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u^2 + \mathrm{v}^2)\, d\varkappa dy \; =$$

$$\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (UU^* + VV^*)\, d\omega_1 d\omega_2 \; \ldots \ldots.(4.4.9)$$

so, substituting for (U) and (V) and their complex conjugates from the last two of the equations (4.4.7), we obtain the expression

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2ZZ^*}{\omega_1^2 + \omega_2^2}\, d\omega_1 d\omega_2 \ldots \ldots \ldots..(4.4.10)$$

This integral (E) has a simple physical interpretation – it represents the kinetic energy of the fluid flow. If we from the spectral decomposition of this function , we write

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(\omega_1, \omega_2, t)\, d\omega_1 d\omega_2 \ldots\ldots\ldots..(4.4.11)$$

Where the function $r(\omega_1, \omega_2, t)$ is called the spectral function. A simple expression for this spectral function is obtained by identifying equations (4.4.10) and (4.4.11). We find that [1]

$$r(\omega_1, \omega_2, t) = \frac{2}{\omega_1^2 + \omega_2^2} \; |Z(\omega_1, \omega_2, t)|^2 \ldots..(4.4.12)$$

These equations show that the harmonic analysis of the flow of a viscous fluid can be based on the study of the Fourier transform $Z(\omega_1, \omega_2, t)$ of the vorticity, but even in the rather special case we are considering here, in which the fluid fills the entire space, we have no simple method of determining the form of the function $Z(\omega_1, \omega_2, t)$. We shall now set up the nonlinear integrodifferential equation upon whose solutions the determination of $Z(\omega_1, \omega_2, t)$ rests. From equation (4.4.5) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\varkappa, y)\frac{\partial \xi}{\partial \varkappa}\, e^{i(\omega_1 \varkappa + \omega_2 y)}\, dxdy \; =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -iU(\theta_1, \theta_2)(\theta_1 - \omega_1)Z^*(\theta_1 - \omega_1, \theta_2 - \omega_2)\, d\theta_1 d\theta_2 \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots\ldots(4.4.13)$$

Now the Fourier transform $U(\theta_1, \theta_2)$ is, by equation (4.4.7) of the form

$$\frac{2i\theta_2}{\theta_1^2 + \theta_2^2} Z(\theta_1, \theta_2, t)$$

so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \frac{\partial \xi}{\partial \varkappa} e^{i(\omega_1 \varkappa + \omega_2 y)}\, dxdy =$$

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\theta_2(\theta_1 - \omega_1)}{\theta_1^2 + \theta_2^2} Z(\theta_1, \theta_2) Z^*(\theta_1 - \omega_1, \theta_2 - \omega_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \ldots$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (4.4.14)$$

Similarly it may be shown that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{v}(\varkappa, y) \frac{\partial \xi}{\partial y} e^{i(\omega_1 \varkappa + \omega_2 y)}\, dxdy =$$

$$-2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\theta_1(\theta_2 - \omega_2)}{\theta_1^2 + \theta_2^2} Z(\theta_1, \theta_2) Z^*(\theta_1 - \omega_1, \theta_2 - \omega_2) dxdy \ldots$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots .. \ldots \ldots \ldots (4.4.15)$$

Hence it follows that equation

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial \varkappa} + \mathrm{v}\frac{\partial \xi}{\partial y} = V\nabla_1^2 \xi$$

is equivalent to

$$\frac{\partial Z(\omega_1, \omega_2, t)}{\partial t} + V(\omega_1^2 + \omega_2^2) Z(\omega_1, \omega_2, t) =$$

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\theta_2\omega_1-\theta_1\omega_2)}{\theta_1^2+\theta_2^2}Z(\theta_1,\theta_2,t)Z^*(\theta_1-\omega_1,\theta_2-\omega_2,\mathrm{t})\mathrm{d}\theta_1\mathrm{d}\theta_2\ \ldots..(4.4.16)$$

It will be observed that, in the case in which $(\theta_2\omega_1-\theta_1\omega_2=0)$, this equation reduces to the approximate equation

$(\frac{\partial\zeta}{\partial t}=-V(\omega_1^2+\omega_2^2)Z(\omega_1,\omega_2,t))$ which we used previously to obtain the solution of special problems. This suggests that in a case in which the term on the right – hand side of equation (4.4.16) would appear to be small we may obtain approximate solutions of the equation by writing ($Z = Z_0 + Z_1$), where Z_0 is the appropriate solution of the corresponding initial value problem on the equation $(\frac{\partial\zeta}{\partial t}=-V(\omega_1^2+\omega_2^2)Z(\omega_1,\omega_2,t))$. If we then insert this form in equation (4.4.16) and retain only terms in (Z_0) on the right, we obtain the equation

$$\frac{\partial Z_1(\omega_1,\omega_2,t)}{\partial t}=V(\omega_1^2+\omega_2^2)Z_1(\omega_1,\omega_2,t)$$

$$=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\theta_2\omega_1-\theta_1\omega_2)}{\theta_1^2+\theta_2^2}Z_0(\theta_1,\theta_2,t)Z_0^*(\theta_1-\omega_1,\theta_2-\omega_2,\mathrm{t})d\theta_1 d\theta_2\ \ldots\ldots\ldots\ldots..(4.4.17)$$

for the determination of the small quantity (Z_1).[1]

This equation being linear in (Z_1), may be solved readily under the initial condition ($Z_1 = 0$) to give the first approximation to (Z_1). The process may then be repeated, retaining terms of the next highest order of small quantities on the right. In this way we may build up the solution in cases in which the nonlinear terms represent a small perturbation of the linear flow described by equation ($\frac{\partial \xi}{\partial t} = V\nabla_1^2 \xi$). In the general case in which the nonlinear term is appreciable there does not appear to be any general method of attack. [1]

In the above analysis we have assumed that the viscous fluid fills the whole of the ($\varkappa$ y) plane. If it is bounded by a curve (C) which encloses a domain (D) of finite area (S), then we may employ finite Fourier transforms of the type.

$$F(\omega_1, \omega_2) = \frac{1}{2\pi} \int\int_D f(\varkappa, y)\, e^{-i(\omega_1 \varkappa + \omega_2 y)}\, dxdy \ldots (4.4.18)$$

to obtain some general inequalities governing the behavior of certain physical quantities. It is obvious that

$$|\mathrm{F}(\omega_1, \omega_2)| \leq \frac{1}{2\pi} \int\int_D |f(\varkappa, y)\,| d\varkappa dy \quad \ldots\ldots (4.4.19)$$

so that, by schuarz's inequality, we have

$$|\mathrm{F}(\omega_1, \omega_2)|^2 \leq \frac{S}{4\pi^2} \left[\int\int_D f^2(\varkappa, y)\, d\varkappa dy\right] \quad \ldots. (4.4.20)$$

since $S = \int\int_D \, d\varkappa dy$. Extracting the square root, we obtain

$$\mathrm{F}(\omega_1, \omega_2) \leq \frac{S^{\frac{1}{2}}}{2\pi} \left[\int\int_D f^2(\varkappa, y)\, d\varkappa dy\right]^{\frac{1}{2}} \ldots\ldots (4.4.21)$$

In particular, if (U) denotes the finite transform of u$(\varkappa, y)$, then

$$|\mathrm{U}| \leq \frac{S^{\frac{1}{2}}}{2\pi} \left(\int\int_D U^2\, d\varkappa dy\right)^{\frac{1}{2}} \leq \frac{(2SE)^{\frac{1}{2}}}{2\pi} \quad \ldots \ldots (4.4.22)$$

Where (E) is the total energy of the fluid, defined by the analogue of equation (4.4.9). Similarly

$$|V| \leq \frac{(2SE)^{\frac{1}{2}}}{2\pi} \quad \dots\dots\dots\dots\dots\dots\dots (4.4.23)$$

Now from equation (4.4.8) it follows that

$$2iZ(\omega_1, \omega_2) = \omega_2 U - \omega_1 V \dots\dots\dots . (4.4.24)$$

and hence

$$|Z(\omega_1, \omega_2)| \leq \frac{1}{2}(|\omega_2| \, |U| + |\omega_1| \, |V|) \dots .. (4.4.25)$$

Substituting from equations (4.4.22) and (4.4.23) into equation (4.4.25), we find that

$$|Z(\omega_1, \omega_2)| \leq \frac{(2SE)^{\frac{1}{2}}}{4\pi} (|\omega_1| + |\omega_2|) \dots .. (4.4.26)$$

Equation (4.4.12) then gives and upper bound for $\gamma(\omega_1, \omega_2, t)$,

$$\gamma(\omega_1, \omega_2) \leq \frac{SE}{4\pi^2} \frac{(|\omega_1| + |\omega_2|)^2}{\omega_1^2 + \omega_2^2}$$

From which it follows immediately that

$$\gamma(\omega_1, \omega_2) \leq \frac{SE}{2\pi^2} .$$

Indeed it can be proved that [1]

$$\gamma(\omega_1 , \omega_2) \leq \frac{SE_0}{2\pi^2} \, e^{-8vt/kD^{\frac{1}{2}}}$$

(4-5) Stability of theory of Hydrodynamic

We assume that at every point x of the fluid, and at all times t, we can define properties like density ρ(x, t), velocity u(x, t), and pressure p(x, t), and that these vary smooy (differentiably) over the fluid. Note that we do not deal with the dynamics of individual molecules. A small volume δV thus has mass δVρ and momentum δV ρu. [3]

The material derivative: A fluid particle, sometimes called a material element, is one that moves with the fluid, so that its velocity is u(x, t) and its position x(t) satisfies $\dot{x}$ = u(x, t). The rate of change of a quantity as seen by a fluid particle is called the material derivative and written D/Dt. It is given by the chain rule as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u.\nabla \;\ldots\ldots\ldots\ldots\ldots\ldots\; (4.5.1)$$

Mass conservation:

$$\frac{D\rho}{Dt} + \rho\nabla.u = 0 \ldots\ldots\ldots\ldots..(4.5.2)$$

For an incompressible fluid, the density of each material element is constant, and Incompressible flow:

$$\frac{D\rho}{Dt} = 0 \quad , \quad \nabla.u = 0 \ldots\ldots\ldots...(4.5.3)$$

In this Reserch we shall concentrate on fluids that are incompressible and have uniform density, so that ρ is independent of both x and t. [3]

Streamfunctions in 2D and axisymmetry . For two-dimensional flows, the condition ρ· u = 0 is automatically satisfied by

$$u = \nabla\wedge(0,0,\varphi(x,y)) = (\varphi_{y},_{\varphi_x},0) \ldots\ldots\ldots..(4.5.4)$$

ɸ(x, y) is called the streamfunction.

In axisymmetric flows, in terms of cylindrical polar coordinates (r, θ, z), the incompressibility condition $\nabla\cdot u = 0$ is satisfied using the Stokes streamfunction, $\varphi\,(r, z)$,

$$u = \nabla\wedge\left(0,\frac{\varphi}{r},0\right) = \left(-\frac{1}{r}\frac{\delta\varphi}{\delta z},0,\frac{1}{r}\frac{\delta\varphi}{\delta r}\right)\ldots\ldots\ldots..(4.5.5)$$

The Navier-Stokes Equations for an incompressible fluid

$$\rho \frac{Du}{Dt} \equiv \rho \left(\frac{\delta u}{\delta t} + (u \,.\, \nabla)\, u \right) = -\nabla_{\rho} + F + \mu \nabla^2 u \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.5.6)$$

$$\nabla \,.\, u = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (4.5.7)$$

In (4.5.6) µ is the viscosity, assumed constant, and F a body force, perhaps gravity, F = ρg. In cylindrical polar cordinates, (r, θ, z), with velocity u = (u_r , u_θ , u_Z), (4.5.6 - 4,5.7) become Cylindrical

$$\rho \left(\frac{Du_r}{Dt} - \frac{u\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right)$$

Polar

$$\rho \left(\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu (\nabla^2 u_\theta - \frac{u_\theta}{r^2} - \frac{2}{r^2} \frac{\partial u_r}{\partial \theta}$$

Coordinates

$$\rho \frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + \mu \nabla^2 u_z$$

$$\frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_\theta}{\partial z} = 0 \ldots\ldots\ldots\ldots (4.5.8)$$

Where

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u_r \frac{\partial f}{\partial r} + u_\theta \frac{\partial f}{\partial \theta} + u_z \frac{\partial f}{\partial z}$$

and

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Boundary Conditions. [3]

In order to determine the velocity u(x, t) and pressure p(x, t) in some region V , we need to know what boundary conditions to apply on the surface S. The appropriate conditions to apply are that the velocity and the total stress should be continuous across any interface. Here 'total stress' includes any surface tension (see below.)

(a) Fluid/solid boundaries:

A solid boundary can provide whatever stress is needed to support the fluid motion, so it is sufficient to require that the fluid velocity u be the same as the velocity of the boundary. Thus for a stationary boundary

u = 0 . ……………………………….(4.5.9)

Note that requires that the tangential velocity components be zero as well as the normal component. In inviscid flow only the normal velocity need be continuous at an interface, and a 'slip velocity' must be permitted. The presence in the Navier-Stokes equation of the second derivative $\mu\nabla^2$u requires an extra boundary condition.

(b) Fluid/fluid boundaries:

These are more complicated, because the interface can move. Furthermore, it is a physical fact that an extra normal stress, due to surface tension, acts on the interface. This extra stress takes the form γK(x) where γ is the positive surface tension constant, and K is the curvature of the fluid surface, which can be defined by K = $\nabla\cdot\hat{n}$ where $\hat{n}$ is the unit normal to the interface. [3]

If one of the fluids is dynamically negligible, as often happens with a liquid/gas interface, then we can treat one fluid as having a constant pressure p_0 and neglect its motion. If the interface is stationary, then the appropriate boundary conditions to apply on the other fluid are zero normal velocity and zero tangential stress. (So if the surface is y = 0 and velocity (u, v, 0) then we have v = 0 and μ $\partial u/\partial y$ = 0. For inviscid flow, μ = 0 and the tangential stress condition is trivial.) If the interface moves and we describe its position at time t by the function ζ(x, t) = 0, then the kinematic boundary condition for the normal velocity can be written

$$\frac{D\zeta}{Dt} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4.5.10)$$

Inviscid and high-Reynolds-number Flows .

When written in terms of nondimensional variables, a parameter, Re, known as the Reynolds number appears in the equations. Re essentially measures the relative importance of the inertial to the viscous forces. and is defined by Re = ρLU/μ where L is a typical length-scale of the problem, and U a typical velocity magnitude.

At low values of Re, it can be proved that only one steady solution of the Navier-Stokes equations exists, and that this flow is stable in the sense defined below. For high values of Re, there are many examples where more than one stable, steady solution is known to exist. Flow instability is strongly linked with the existence of more than one solution.

When Re ≫ 1, it is tempting to neglect the viscous terms, setting μ = 0. If this is done, one of the boundary conditions must be omitted, usually allowing tangential slip. Some caution is necessary, as viscous boundary layers form near solid surfaces in which the velocity develops strong gradients so that the viscous term cannot be neglected. Boundary layers typically have thickness $\frac{L}{R_e^{\frac{1}{2}}}$ and must remain thin for the "core/layer" structure to be valid. Inside a steady boundary layer, where x and y are measured parallel and normal to the boundary, the governing equations for u = (u, v, 0)

$$\rho\left(uu_x + vv_y\right) = -P_x + \mu u_{yy}\,, P_y = 0$$

$$u_x + v_y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4.5.11)$$

These equations are parabolic which means they must be solved in the downstream direction. The pressure does not vary across the layer and is determined by the conditions at "y = ∞" which means the external potential flow. The boundary layer equations tend to be valid so long as the pressure gradient is favourable, which means $-p_x > 0$. If the pressure gradient is unfavourable, there is a strong likelihood that separation of the boundary layer will occur. This is manifested by the solution to the boundary layer equations developing a singularity. Separation completely alters the external flow, and leads for example to "stall" of aircraft. [3]

The vorticity equation is obtained by taking the curl of (4.5.6). Writing $\omega = \nabla$ ^ u we have

$$\rho\left(\frac{\partial\omega}{\partial t} + u.\nabla\omega - \omega.\nabla u\right) = \mu\nabla^2\omega \text{ ----------- } (4.5.12)$$

For two-dimensional flow, if we write u = ∇^ (0, 0, ψ (x, y, t)) and ω= (0, 0, ω) then

$$\rho\frac{D\omega}{Dt} = \mu\nabla^2\omega\ ,\ and\ \omega = -\nabla^2\psi \ldots\ldots\ldots..\ (4\,.5\,.13)$$

A flow for which ω= 0 everywhere is said to be irrotational. Then we can introduce a velocity potential, ∅, such that u = ∇∅.

Inviscid Flows: As there is no source term in (4 . 5 . 12), vorticity can only be generated at boundaries. If μ = 0 then a flow which is irrotational initially remains irrotational for all time. The time-dependent Bernoulli theorem states that for irrotational flows,

$$\frac{\partial \emptyset}{\partial t} + \frac{1}{2}|\nabla \emptyset|^2 + gy + \frac{P}{\rho} = constant --- (4.5.14)$$

Turbulence: At high values of R_e it is found experimentally that fluid flows tend to become unsteady and highly chaotic, even though a simple steady flow could exist in theory. Turbulent flows are difficult to analyse and have important practical implications. The manner in which transition to turbulence of a laminar flow occurs is an important topic. The first stage in this process is that the underlying steady flow becomes unstable. In this course we examine Hydrodynamic Stability.[3]

Stability Concepts

For a given problem, we solve the governing equations and obtain a solution which we assume is steady, u = U(x) with a corresponding pressure distribution p = P(x). We then make a small perturbation to the flow, so that

$$u = U(x) + \varepsilon\acute{u}(x,t), \; p = P(x) + \varepsilon\acute{p}(x,t) \ldots\ldots(4.5.15)$$

where ε is a small positive constant. We then consider the behaviour of $\acute{u}$. If $\varepsilon\acute{u}$ remains small for all time, we say that the underlying flow is stable, whereas if it eventually becomes large no matter how small ε is, we say the flow is unstable.

The exact equations for $\acute{u}$ and $\acute{p}$ are

$$\rho\left(\frac{\partial \acute{u}}{\partial t} + U.\nabla\acute{u} + \acute{u}.\nabla U + \varepsilon\acute{u}.\nabla\acute{u}\right) = -\nabla\acute{p} + \mu\nabla^2\acute{u}$$

$$-\nabla.\acute{u} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.5.16)$$

Linear stability theory neglects the last term on the LHS, as ε is arbitrarily small. The resulting linear equation has solutions of the form $\acute{u} = \hat{u}(x)e^{st}$ for some vector function $\hat{u}$ and constant s, and $similarly\, \acute{p} = \hat{p}(x)e^{st}$. This is because none of the coefficients depends on t as U is steady. The general solution to this problem will be a linear combination of all these particular solutions. The possible values of s can be regarded as eigenvalues of the system. These can be real, but are in general complex

$$s = s_r + is_i \ , e^{st} = e^{s_r t}[\cos(s_i t) + i \sin(s_i t)]$$

(a) If for all possible values of s we have $s_r < 0$ we say the flow is stable.
(b) If there is at least one eigenvalue s for which $s_r > 0$, the flow is unstable.
(c) If $s_r = 0$ for some eigenvalue, we say the flow is neutrally stable. In this case nonlinear terms may be particularly important.
Surface stability: If the fluid has a free surface, this will deform in accordance with the normal stress associated with the perturbation velocity Free surfaces can be unstable even at very low R_e. [3]

The above approach looks at perturbation modes with a fixed spatial structure and examines how they evolve in time, a process known as temporal stability. An alternative approach, which is often appropriate, is to consider the spatial evolution of a localized disturbance in the flow. This disturbance may grow as it is advected downstream, so that the place where the instability occurs is far away from the disturbance. This is known as convective instability. In practice it is possible that the region of flow interest is too small for an instability of a given initial magnitude to develop. If a disturbance at a given position leads to growth at that position this is known as absolute instability. [3]

Condusions

The Hydrodynamic mathematical problems is for system of equation .

The different problems in fluid mechanics yield different system of addition conditions which must be imposed on the solution of the Navier – stoke equation . Since mathematical problems in fluid mechanics and the motion may depend certain problems .

References

1- IANN – SNEDDON , Fourier Transforms , New York ,TORONTO LONDON , MCGRAW.HILL BOOK COMPANY , INC 1951.

2- D . H . WILSON , M . SC , Hydrodynamics , Lecturer in Applied Mathematics , The Durham Colleges in The University of Durham 1959.

3- Drazin . P . G .Introduction to hydrodynamics stability , Cambridge University prees 2002.

4- JOHN M CIMBALA ,FLUID MECHANICS ,Fundamentals and Applications YUNUSA . CENGEL JOHN M CIMBALA 2006

5- Adelin Georgescu , Hydrodynamic stability theory Nijhoff 1985 .

6- Lamb.S . H , HYDRODYNAMICS 6th edition , Dover publication 1993 .

7- http:// WWW . google – com / search ? g :