Linear Analysis of an Orthotropic Beam

M. Y. Osman*

Abstract

An approximate small deflection solution of a simply supported orthotropic beam is given. The beam is treated as a plane stress problem. Using the elasticity relations for the material under consideration, a second-order differential equation of the longitudinal displacement is obtained. The solution is found with the bending moment expressed as a Fourier series. The deflection and stresses obtained are compared with those predicted by the simple beam theory. Reasonable agreement is found between the transverse shearing stresses. However the deflections and longitudinal stresses according to the present theory - and in particular deflections - can be considerably greater than those given by the simple beam theory. Correlations between deflections and stresses are mathematically expressed in terms of aspect ratio and the degree of orthotropy of the beam.

*Nile Valley University

Notation

b, h, L	breadth, depth, and length of beam respectively
C_1 , C_2	stiffnesses
E_x , G_{xz}	longitudinal and shear moduli respectively
K	shear correction factor
u, w	longitudinal and transverse displacements respectively
x, z	beam co–ordinate system
q, p	load per unit area and load per unit length respectively
ϵ_x, ϵ_{xz}	longitudinal and shear strains respectively
ν	Poisson's ratio
σ_x, σ_{xz}	longitudinal and shear stresses respectively
$\lambda \left(= \frac{L}{h}\right)$	aspect ratio
$R \left(= \frac{E_x}{G_{xz}}\right)$	degree of orthotropy

1 Introduction

When bending of a beam is caused by shear forces, and if the deflection, w, is small and the shear stress on any transverse section is assumed constant through thickness and rapidly vanishes on top and bottom of the beam such that plane cross–sections before bending remain plane but not necessarily normal to the mid–plane of the beam as in the simple beam theory, the curvature of a beam which is loaded uniformly throughout can be expressed as follows:

$$\frac{d^2w}{dz^2} = \frac{1}{EI}\left(M + \frac{EI}{KAG}p\right)$$

Where E is Young's modulus, I is second moment of area, M is bending moment, G is shear modulus, p is load per unit length, and K is shear correction factor introduced as a result of the simplification that made the shear stress constant across thickness rather than parabolic as it is in real.

The first term on r.h.s. of the above equation is due to pure bending and the second term is due to shear. The latter term becomes significant only when the aspect ratio of the beam is small and/or the degree of orthotropy is large.

Considerable work has been done in the area of the analysis of beams and in particular evaluation of shear correction factor. The simplest form of the shear correction factor is suggested by Timoshenko [1] obtained by defining K as the ratio of the average shear strain to the shear strain at the centriod i.e $K = 2/3$. Generally the shear factor for isotropic materials is given in terms of the Poisson's ratio. See for instance Stephen [2].

Shear deformation in composites is more significant than in metals because of their orthotropic nature reflected by their high ratio of longitudinal modulus to shear modulus.

Two approaches have been used in the analysis of composites. In the first approach, the theories developed originally for conventional isotropic materials were extended to composites, and shear correction factor have been estimated as in Sangiahnadar and McGutchen [3]. In the second approach new analytical methods were used and a few assumptions were incoporated to facilitate the solution. See for instance Everstine and Pibkin [4].

In the study of shear factor, it is usually assumed that the distortion of transverse cross–sections due to shearing forces does not affect the evaluation of longitudinal and shear stresses as in Gay [5] and Markenscoff [6]. In many cases, it is assumed that the simple beam theory stress pattern prevails i.e. that the longitudinal stress is linear, the shear stress across thickness is parabolic, and transverse stress is zero. Although the effect of shear deformation on longitudinal stresses in isotropic beams may be negligible, it is not equally true for composite beams as this work explains.

In this paper attention is paid to composites which are known to be orthotropic designed such that the fibres become the prime load carriers.

2 The beam equations

2.1 The strain–displacement relations

The assumed displacements are:

$$\begin{aligned} u &= u(x,z) \\ w &= w(x) \end{aligned}$$

See Fig.1 for the co–ordinate system. The strain–displacement relations are then:

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{1}$$

The solution is then written in the form:

$$u = \sum_{n=1}^{\infty} U_n cos(\alpha x) sinh\left(\frac{\alpha z}{\beta}\right) \tag{9}$$

where U_n is a constant and $\alpha = n\pi/L$.
Substitute in Eqn(6) to obtain:

$$\sigma_x = -C_1 \sum_{n=1}^{\infty} \alpha U_n sin(\alpha x) sinh\left(\frac{\alpha z}{\beta}\right) \tag{10}$$

The boundary condition for σ_x is as follows:

$$\int_{h/2}^{-h/2} \sigma_x z dz = \sum_{n=1}^{\infty} M_n sin(\alpha x)$$

where the term on the rhs of the above equation represents the bending moment, and M_n is a constant. Hence:

$$U_n = -M_n / \left\{ C_1 \left[\beta h cosh\left(\frac{\alpha h}{2\beta}\right) - \frac{2\beta^2}{\alpha} sinh\left(\frac{\alpha h}{2\beta}\right)\right]\right\} \tag{11}$$

Substitute Eqn(9) in Eqn(5) and integrate w.r.t. z between the limits $(-h/2)$ and $(h/2)$ to obtain:

$$C_2 \left\{ \frac{\partial}{\partial x}[U]_{h/2}^{-h/2} + h\frac{d^2 w}{dx^2} \right\} + q = 0$$

where $q = \sigma_z\left(\frac{-h}{2}\right) - \sigma_z\left(\frac{h}{2}\right)$
After some manipulation, we obtain:

$$\frac{d^2 w}{dx^2} = \sum_{n=1}^{\infty} \frac{\alpha^2}{\beta} U_n cosh\left(\frac{\alpha h}{2\beta}\right) sin(\alpha x)$$

When the beam is simply supported at its ends, the deflection is given as follows:

$$w = -\sum_{n=1}^{\infty} \frac{U_n}{\beta} cosh\left(\frac{\alpha h}{2\beta}\right) sin(\alpha x) \tag{12}$$

Substitute Eqns(11 & 12) in Eqn(7) to obtain:

$$\sigma_{xz} = -C_2 \sum_{n=1}^{\infty} \frac{U_n \alpha}{\beta} \left\{ cosh\left(\frac{\alpha h}{2\beta}\right) - cosh\left(\frac{\alpha z}{\beta}\right)\right\} cos(\alpha x) \tag{13}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \tag{2}$$

$$\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{dw}{dx} \tag{3}$$

2.2 The equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{4}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{5}$$

2.3 The constitutive equations

$$\sigma_x = C_1 \epsilon_x \tag{6}$$

$$\sigma_{xz} = C_2 \epsilon_{xz} \tag{7}$$

where $C_1 = E_x/(1 - \nu_{xz}\nu_{zx})$ and $C_2 = G_{xz}$.

3 The analytical solution

Substitue Eqns(6&7) in Eqn(4) to obtain:

$$\frac{\partial^2 u}{\partial x^2} + \beta^2 \frac{\partial^2 u}{\partial z^2} = 0 \tag{8}$$

where :

$$\beta^2 = \frac{C_2}{C_1} = \frac{G_{xz}}{E_x}(1 - \nu_{xz}\nu_{zx})$$

We seek a solution for Eqn(8) in the form: $u = X(x)Z(z)$ or:

$$u = [Acos(\alpha x) + Bsin(\alpha x)] \left[Ce^{\alpha z/\beta} + De^{-\alpha z/\beta}\right]$$

where $A, B, C, D,$ and α are constants.

For an orthotrpic simply supported beam of length L

$$u(x, 0) = \frac{\partial u}{\partial x}(0, z) = \frac{\partial u}{\partial x}(L, z) = 0$$

3.1 The non-dimensional beam equations

It is convenient to introduce the following non-dimentional quantities:

$$x' = \frac{x}{L}, \quad (h', w', b', z') = \frac{1}{h}(1, w, b, z), \quad \alpha' = L\alpha, \quad u' = \left(\frac{L}{h^2}\right)u,$$
$$\epsilon' = \left(\frac{L}{h}\right)^2 \epsilon, \quad M' = \left(\frac{L^2}{E_x h^4}\right)M, \quad q' = \left(\frac{L^4}{E_x h^4}\right)q, \quad p' = \left(\frac{L^4}{E_x h^5}\right)p,$$
$$\sigma' = \left(\frac{L^2}{E_x h^2}\right)\sigma, \quad C_i' = \frac{1}{E_x}C_i (i = 1, 2)$$

The non-dimensional equations are: (primes are omitted):

$$u = \sum_{n=1}^{\infty} U_n cos(\alpha x) sinh\left(\frac{\alpha z}{\beta\lambda}\right) \tag{14}$$

where:

$$U_n = -M_n / \left\{ C_1 \lambda \left[\beta cosh\left(\frac{\alpha}{2\beta\lambda}\right) - \frac{2\beta^2\lambda}{\alpha} sinh\left(\frac{\alpha}{2\beta\lambda}\right)\right]\right\} \tag{15}$$

$$\sigma_x = -C_1 \sum_{n=1}^{\infty} \alpha U_n sin(\alpha x) sinh\left(\frac{\alpha z}{\beta\lambda}\right) \tag{16}$$

$$\sigma_{xz} = -C_2 \sum_{n=1}^{\infty} \frac{U_n \alpha}{\beta} \left\{ cosh\left(\frac{\alpha}{2\beta\lambda}\right) - cosh\left(\frac{\alpha z}{\beta\lambda}\right)\right\} cos(\alpha x) \tag{17}$$

$$w = -\frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{U_n}{\beta} cosh\left(\frac{\alpha}{2\beta\lambda}\right) sin(\alpha x) \tag{18}$$

For a beam simply supported at its ends having rectangular cross-section and loaded uniformly throughout

$$M_n = \frac{2p}{\alpha^3 b}\{(1 - cos(\alpha)\}$$

where p is the load per unit length.

4 The simple beam theory solution

It can be shown that the deflection and stresses in a beam simply supported at its ends and uniformly loaded throughout according to the simple beam theory are as follows:

$$w = \frac{px}{b}(x^2 - 0.5x^3 - 0.5) \quad (19)$$

$$\sigma_x = \frac{6p}{b}x(1-x)z \quad (20)$$

$$and \quad \sigma_{xz} = \frac{3p}{\lambda b}(1-2x)(0.2-z^2) \quad (21)$$

5 Discussion

In order to assess the accuracy of the theory presented, a beam simply supported at its ends and loaded uniformly throughout is analysed. The cross-section of the beam is rectangular with breadth equals half depth ($b = h/2$). In the analysis it is assumed that the product of the Poisson's ratios $\nu_{xz}\nu_{zx} \ll 1$ and therefore neglected. The aspect ratio of the beam is made to vary between 4 and 20; and the degree of orthotropy varies between 2.5 for an isotropic material to 80 for a highly orthotropic material.The rate of loading is taken as 3 so that the centre deflection according to the simple beam theory is in the order of the depth of the beam. This may not be practical, but nevertheless serves to demonstrate that deflection and longitudinal stress are both dependent to different extents on the aspect ratio and degree of othotropy of the beam.

The centre deflections are compared with Ref[7] in Table (1). The maximum difference for a very short beam ($\lambda = 4$) and highly orthotropic material ($R = 80$) is about 6%. See Table (1).

Table (2) gives the centre deflection for the whole range of aspect ratio and degrees of orthotropy mentioned earlier. The actual deflection w_c can be up to five times that given by the simple beam theory. The centre deflections (w_c) are plotted against degree of orthotropy (R) in Fig. 2, and against aspect ratio of the beam (λ) in Fig.3. The deflection increases linearly with the degree of orthotropy whereas it decays exponentially with the aspect ratio. It is evident that as the aspect ratio decreases or the degree of orthotropy increases, the deflection graph departs away from the simple beam theory line graph. It can be concluded that when the aspect ratio of the beam $\lambda \geq 20$, the simple beam theory gives reasonably accurate results irrespective of the degree of orthotropy. The percentage difference (ρ_d) between the deflections obtained by this method (w_1) and the simple beam theory (w_2) i.e.:

$$\rho_d = \frac{w_1 - w_2}{w_2} \times 100$$

is given in Table (4) and plotted against the degrees of orthotropy (R) and the aspect raios (λ) in Fig.4. It is possible to suggest a mathematical model for the percentage difference as in Ref[7] in the form:

$$\rho_d = aRe^f \qquad (22)$$

where

$$f = d(1 - \lambda^n)$$

where $a, d,$ and n are constants. In fact it is found that the deflection can easily be obtained by multiplying the deflection predicted by the simple beam theory by a factor, called Zeta deflection–factor, ζ_d. This factor takes the form:

$$\zeta_d = 1 + 0.835Re^f \qquad (23)$$

where

$$f = 95(1 - \lambda^{0.02})$$

It has been verified that this factor is accurate when calculating the deflection at any point along a beam of any rectangular cross–section. That consistent accuarcy suggests that the Zeta deflection factor can be equally accurate for the general case of a beam of any cross–sectional shape and loaded in any manner.

In Table (4) the longitudinal stress at the bottom centre of the beam are given. The results are plotted in Fig.5 and Fig.6. Again as with deflection, the longitudinal stress varies linearly with the degree of orthotropy wheres it decays exponentially with the aspect ratio. As the aspect ratio decreases and the degree of orthotropy increases, the stress graph departs from the simple beam theory line graph. The percentage difference (ρ_s) between the stress obtained by the present theory (σ_{x1}) and that predicted by the simple beam theory (σ_{x2}) i.e.

$$\rho_s = \frac{\sigma_{x1} - \sigma_{x2}}{\sigma_{x2}} \times 100$$

is given in Table (5) and plotted in Fig.7. The ($\rho_s - R$) graph for all values of λ are straight lines. However for $\lambda = 4$, one notices a slight curvature. ($\rho_s - \lambda$) graph is exponential as in the case of deflection. By a similar approach a Zeta stress–factor, ζ_s, is suggested which is a function of the aspect ratio and degree of orthotropy of the beam i.e.

$$\zeta_s = 1 + 0.071 Re^{g} \tag{24}$$

where

$$g = 17.4(1 - \lambda^{0.09})$$

The Zeta stress factor when multiplied by the stress given by the simple beam theory gives the actual stress that agrees with the present theory. The accuracy of the Zeta stress factor has been verified by calculating the longitudinal stress at different points in beams with different rectangular cross–sections. The consistent accuracy suggests that the longitudinal stress in an orthotropic beam with general cross–section and general loading can be accurately calculated using the Zeta stress factor.

The transverse shear stresses are given in Table (6). The percentage differences for the shear stress calculated in a similar way as for deflection and longitudinal stress are given in Table (7). The maximum difference between the shear stress obtained by the present analysis and the simple beam theory for exceptionally highly orthotropic material ($R = 80$) and for exceptionally short beam ($\lambda = 4$) is less than 17%. Generally the difference is very small and therefore it can be concluded that transverse shear stress can be calculated with reasonable accuracy using the simple beam theory.

6 Conclusions

1. The deflection and the longitudinal stress in an orthotropic beam depend on the aspect ratio and degree of orthotropy of the beam analysed.

2. The relation between deflection and degree of orthotropy is linear whereas the relation between deflection and aspect ratio is exponential. Similar relations exist between longitudinal stress, degree of orthotropy, and aspect ratio.

3. The deflection of a short and highly orthotropic beam can be several times greater than that predicted by the simple beam theory. The longitudinal stress in the same beam on the other hand can be more than 50% greater than the stress predicted by the simple beam theory.

4. The deflection and longitudinal stress in an orthotropic beam can be obtained by multiplying the deflection and longitudinal stress due to the simple beam theory by appropriate factors called Zeta deflection–factor and Zeta stress–factor respectively.

5. Mathematical expressions for Zeta deflection factor and Zeta stress factor are given. Both factors are functions of the aspect ratio and degree of orthotropy of the beam analysed.

6. The transverse shear stress in an orthotropic beam can be taken as that predicted by the simple beam theory without appreciable loss in accuracy.

References

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Table 1: Comarison between centre deflection of a simply supported uniformly loaded beam ($p = 3$) and values given in Ref[8].

λ	$R = 2.5$			$R = 80$		
	w_1[a]	w_2[b]	%[c]	w_1	w_2	%
4	1.0759	1.0819	0.6	5.2656	5.6172	6.3
6	0.9980	1.0006	0.3	2.8944	3.0163	4.0
8	0.9707	0.9722	0.2	2.0462	2.1060	2.8
10	0.9581	0.9590	0.1	1.6493	1.6847	2.1
12	0.9512	0.9519	0.1	1.4324	1.4558	1.6
16	0.9444	0.9448	0.0	1.2157	1.2282	1.0
20	0.9412	0.9415	0.0	1.1150	1.1229	0.8

[a] w_1: deflection given by the present theory
[b] w_2: deflection as given in Ref[7]
[c] %: $\frac{w_1 - w_2}{w_2} \times 100$

The centre deflection according to the simple beam theory is 0.9375

Table 2: Centre deflections in a simply supported beam under uniformly distributed load ($p = 3$).

λ	R					
	2.5	10	20	40	60	80
4	1.0759	1.4941	2.0462	3.1347	4.2001	5.2656
6	0.9980	1.1846	1.4324	1.9240	2.4083	2.8944
8	0.9707	1.0759	1.2157	1.4941	1.7692	2.0462
10	0.9581	1.0254	1.1150	1.2938	1.4708	1.6493
12	0.9512	0.9980	1.0603	1.1846	1.3079	1.4324
16	0.9444	0.9707	1.0058	1.0759	1.1454	1.2157
20	0.9412	0.9581	0.9805	1.0254	1.0700	1.1150

Table 3: Percentage difference in centre deflection of a simply supported uniformly loaded beam ($p = 3$) compared with simple theory solution.

λ	R					
	2.5	10	20	40	60	80
4	14.8	59.4	118.3	234.4	348.0	461.7
6	6.5	26.4	52.8	105.2	156.9	208.7
8	3.5	14.8	29.7	59.4	88.7	118.3
10	2.2	9.4	18.9	38.0	56.9	75.9
12	1.5	6.5	13.1	26.4	39.5	52.8
16	0.7	3.5	7.3	14.8	22.2	29.7
20	0.4	2.2	4.6	9.4	14.1	18.9

Table 4: Longitudinal stress at the bottom centre of a simply supported uniformly loaded beam ($p = 3$).

λ	R					
	2.5	10	20	40	60	80
4	4.5870	4.8657	5.2272	5.8986	6.5018	7.0588
6	4.5353	4.6594	4.8246	5.1482	5.4570	5.7554
8	4.5173	4.5870	4.6801	4.8657	5.0473	5.2272
10	4.5090	4.5535	4.6131	4.7323	4.8502	4.9684
12	4.5045	4.5353	4.5766	4.6594	4.7417	4.8246
16	4.5000	4.5173	4.5405	4.5870	4.6333	4.6801
20	4.4979	4.5090	4.5238	4.5535	4.5831	4.6131

Table 5: Percentage difference in longitudinal stress at the bottom centre of a simply supported uniformly loaded beam ($p = 3$) compared with simple beam theory soultion.

λ	R					
	2.5	10	20	40	60	80
4	1.9	8.1	16.2	31.1	44.5	56.9
6	0.8	3.5	7.2	14.4	21.3	27.9
8	0.4	1.9	4.0	8.1	12.2	16.2
10	0.2	1.2	2.5	5.2	7.8	10.4
12	0.1	0.8	1.7	3.5	5.4	7.2
16	0.0	0.4	0.9	1.9	3.0	4.0
20	0.0	0.2	0.5	1.2	1.9	2.5

Table 6: Maximum shear stress at the end of a simply supported uniformly loaded beam ($p = 3$).

λ	R					
	2.5	10	20	40	60	80
4	1.0762	1.0444	1.0180	0.9813	0.9550	0.9344
6	0.7243	0.7104	0.6987	0.6822	0.6695	0.6589
8	0.5458	0.5381	0.5345	0.5222	0.5151	0.5090
10	0.4377	0.4330	0.4288	0.4229	0.4183	0.4144
12	0.3654	0.3622	0.3593	0.3552	0.3521	0.3493
16	0.2745	0.2729	0.2713	0.2690	0.2673	0.2658
20	0.2198	0.2189	0.2179	0.2165	0.2154	0.2144

Table 7: Percentage difference in maximum shear stress at the end of a simply supported uniformly loaded beam ($p = 3$) compared with simple beam theory solution.

λ	R					
	2.5	10	20	40	60	80
4	4.3	7.2	9.5	12.8	15.1	16.9
6	3.4	5.3	6.8	9.0	10.7	12.2
8	3.0	4.3	5.5	7.2	8.4	9.5
10	2.7	3.8	4.7	6.0	7.0	7.9
12	2.6	3.4	4.2	4.9	6.1	6.9
16	2.4	3.0	3.6	4.4	5.0	5.5
20	2.3	2.7	3.2	3.8	4.3	4.7

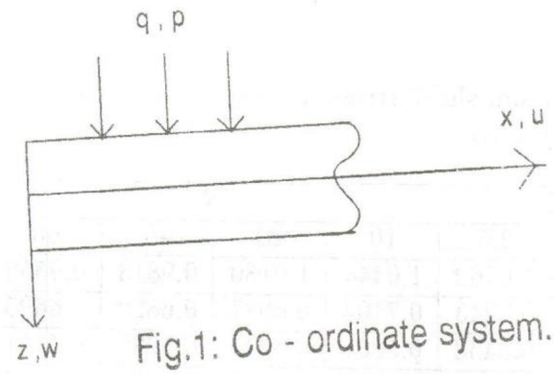

Fig.1: Co - ordinate system.

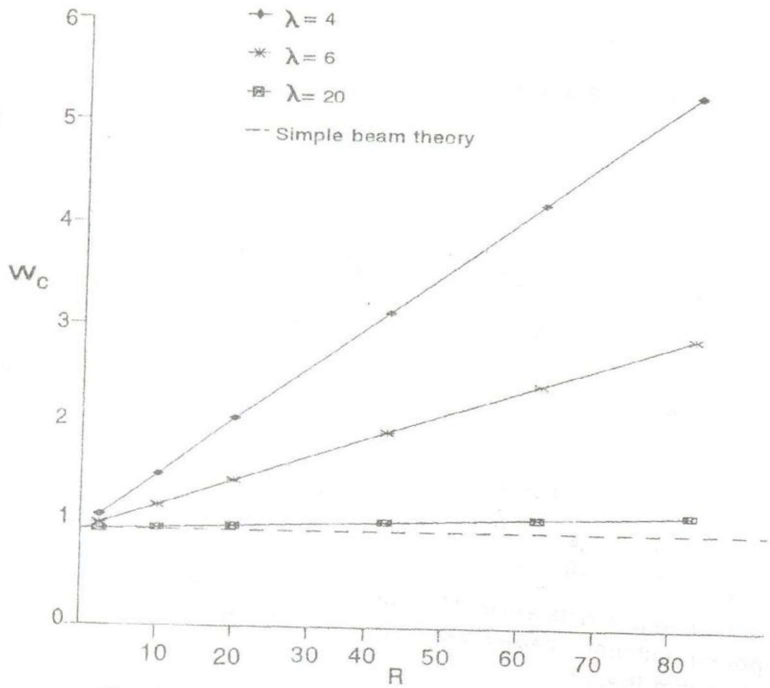

Fig. 2: Centre deflection of a simply supported uniformly loaded beam (p=3) versus degree of orthotropy.

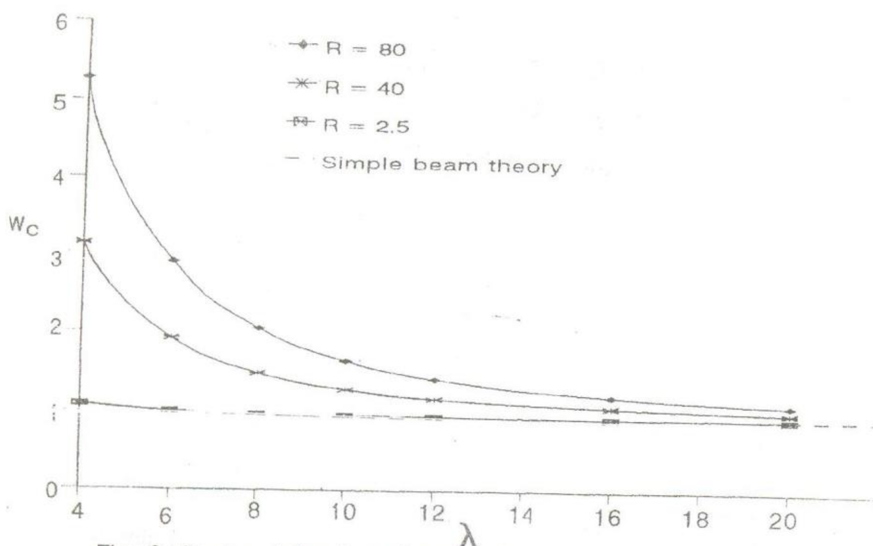

Fig. 3: Centre deflection of a simply supported uniformly loaded beam (p=3) versus aspect ratio.

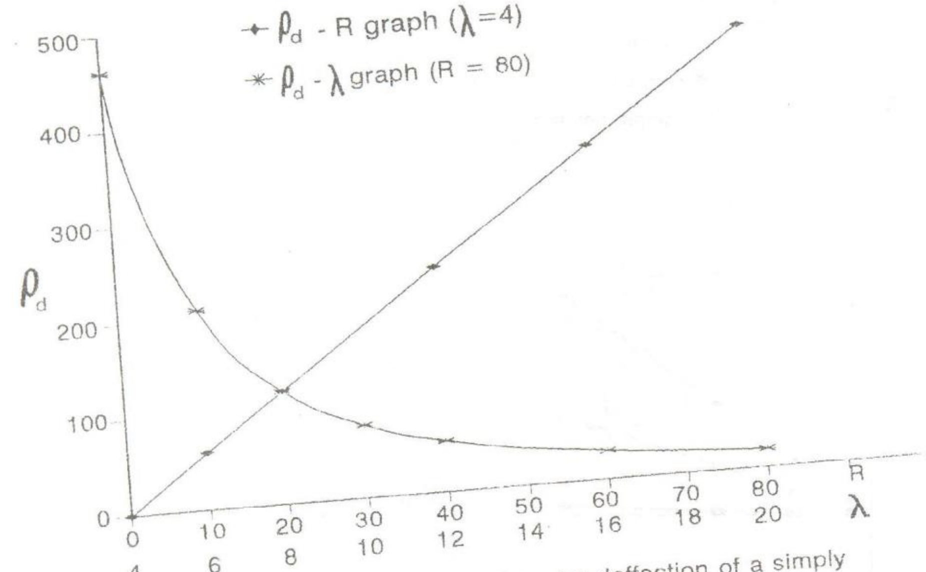

Fig.4: Percentage difference of centre deffection of a simply supported uniformly loaded beam (p = 3) compared with simple beam theory.

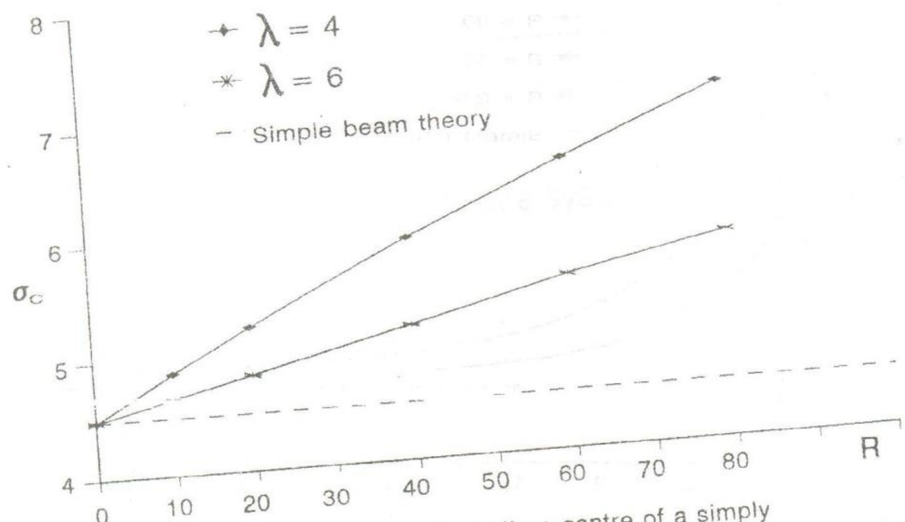

Fig.5: Longitudinal stress at the bottom centre of a simply supported uniformly loaded beam (p=3)versus degree of orthotropy.

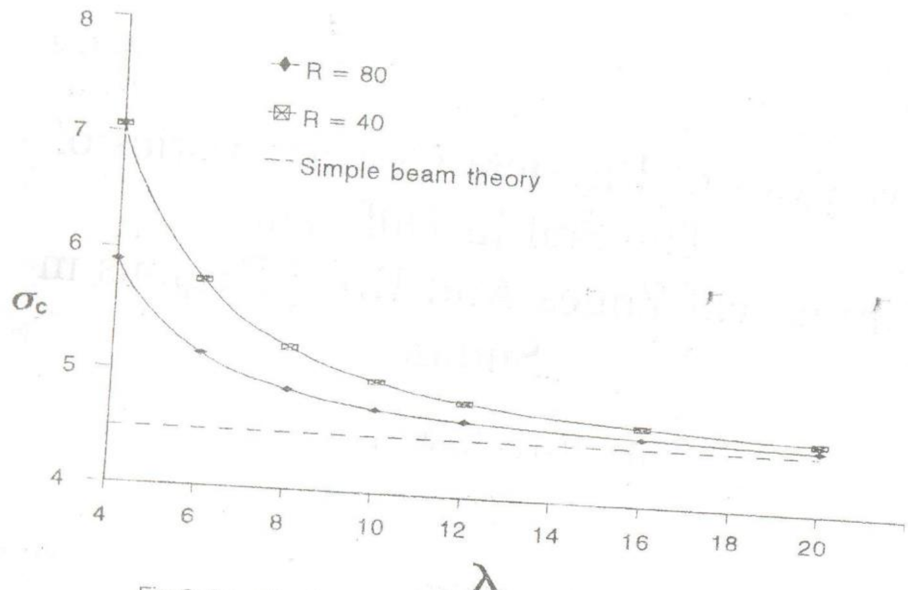

Fig.6: Longitudinal stress at the bottom centre of a simply supported uniformly loaded beam (p = 3) versus aspect ratio.

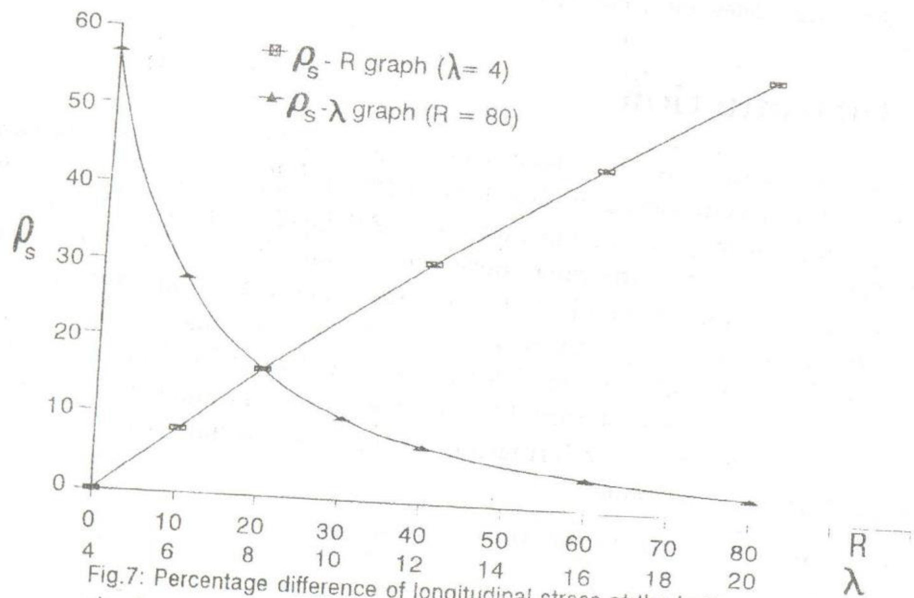

Fig.7: Percentage difference of longitudinal stress at the bottom centre of a simply supported uniformly loaded beam (p = 3) compared with simple beam theory solution.