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COLLAGE OF POST GRADUATE

The analysis of Linear and
Nonlinear Oscillator Systems with periodic orbits

Athesis submitted in partial fulfillment For the Degree of M.Sc. In Mathematics

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Dedication

To my beloved family with love

Acknowledgments

I would like to express my deeply thanks Dr. Adam Abdalla Abbaker, Department of Mathematics, Nile valley university for his supervision, guidance, constructive criticism and changing my ideas through this research project.

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And above all my special praise and unlimited thanks to Alla.

Table of Content

Abstract

In this study we introduced the analysis of some linear and nonlinear oscillator systems , we study the dynamic systems and their periodicity orbits and also we study the local bifurcations of vector fields and maps, and a significant role in the behavior of nonlinear systems. This study discussed four chapters.

In chapter one we introduced the analysis of oscillating systems with one degree of freedom, a model for flow of water in a pump, tank, and pipe system. In chapter tow we study an operational analysis of nonlinear dynamical systems, forced vibrations of nonlinear systems. In chapter three we study linear and nonlinear maps and closed orbits, Poincare maps, and forced oscillations and in chapter four we develop the general theory for dealing with bifurcations of fixed points of n-dimensional flows, the center manifold and normal form theorems. We also study the local bifurcations of maps and develop an analogous theory for them, and studied two–species oscillatory system: bifurcation and stability analysis.

ملخص البحث

في هذه الدراسة تناولت بعض التحليلات للنظم الخطيه وغير الخطيه المتذبذبة وقدم التحليل للنظم الديناميكية ومداراتها الدورية وتمت دراسة أنظمة التفرع المحلية للمجال الإتجاهي للدوال ودورها المهم في سلوك النظم غير الخطية. وهذه الدراسة قدمت في أربعة فصول.

في الفصل الأول قدمت النظم المتذبذبة ذات الدرجة الأولى وقدم نموذج لانسياب الماء في المضخات والأنابيب. وفي الفصل الثاني تناولت دراسة التحليل الفعلي للنظم غير الخطية للتذبذبات ذات القوة للنظم غير الخطية. وفي الفصل الثالث تناول البحث الدوال الخطية والغير خطية ومداراتها المغلقة ودوال بوانكارية والذبذبات ذات القوة. والفصل الرابع تناول النظرية العامة المتعلقة بأنظمة التفرغ عند النقاط الحرجة للانسيابات ذات البعد النوني ومركز عديدة الطيات والشكل الناظم. وتناول أيضاً أنظمة التفرع المحلية للدوال وتم تطوير النظرية المناظرة للأشكال الناظمة. ودراسة نظامين لتذبذب التفرع وتحليل الإستقرار.

Proposal of the Research

The analysis of linear and nonlinear Oscillator Systems

Objectives of the Research:

Here the researcher wants to give precise light to affection the natural phenomena which are being and going to obey rules of linear and nonlinear oscillator systems.

Moreover to find out it possible equations and to look for whether have solutions or no.

Dynamically a large part of engineering and physics is concerned with study of oscillations of nonlinear and linear systems.

No doubt that electrical circuits with lumped parameters are of the same form as the equations for mechanical systems.

Significance of the Research:

Many problems in Dynamic system electrics or other scientific fields use linear and nonlinear systems specially those which have vibrating system.

Oscillations of linear and the analysis of nonlinear oscillatory systems are very important in study of many world problems.

Research Hypothesis:

1. Dynamics and electrics problems often use oscillations of linear and analysis of nonlinear systems in terms of equations or differential equations.

Example:

the linear system:

$$\frac{dx}{dx} = Ax,\ x \in IR^n$$

Where A is an n x n matrix.

2. The nonlinear studies the cases which have a form equation $\dot{x} = \boldsymbol{f}(x)$ where $x = x\ (t) \in IR^n$ is a vector valued function of an independent variable (usually time) and $f{:}U \longrightarrow IR^n$ is a smooth function defined on some subset $U \subseteq IR^n$.
3. The nonlinear and linear flows are related via differ morphisms (Sternberg srequires theorem) certain non-resonance conditions among the eigenvalues of DF(x).
4. The parameters are playing a very important role in how to express about the bifurcations.
5. There are relations between the vector fields and the studying of analysis of bifurcations.

Question of the Research:

1. Are that possible to stay Dynamics and electrics circuits problems use oscillations of linear and analysis of nonlinear systems?

2. Is that possible to find theorem which relates nonlinear and linear flows via?

3. Are Natural phenomena being obey rules of oscillations linearity and analysis of nonlinear systems?
4. Are the rules of linear and nonlinear systems give us some additional insights in to nature of oscillated phenomena and vibrated one's?
5. Are parameters important in a bifurcations and how?

Chapter One

Oscillations of linear systems

1.1. Oscillating systems with one Degree of freedom

Let us consider the vibrating systems of Figure (1)system (*a*) represents a mass that is constrained to move in a linear path. It is attached to a spring of spring constant k and is acted upon by a dashpot mechanism that introduces a frictional constraint proportional to the velocity of the mass. The mass has exerted upon it an external force P_o sinωt. By Newton's law we have.

$$M\ddot{x} = kx - R\dot{x} + P_o \sin \omega t \qquad \left\{ \begin{array}{l} \dot{x} = \dfrac{dx}{dt} \\ \ddot{x} = \dfrac{d^2x}{dt^2} \end{array} \right. \qquad (1.1.1)$$

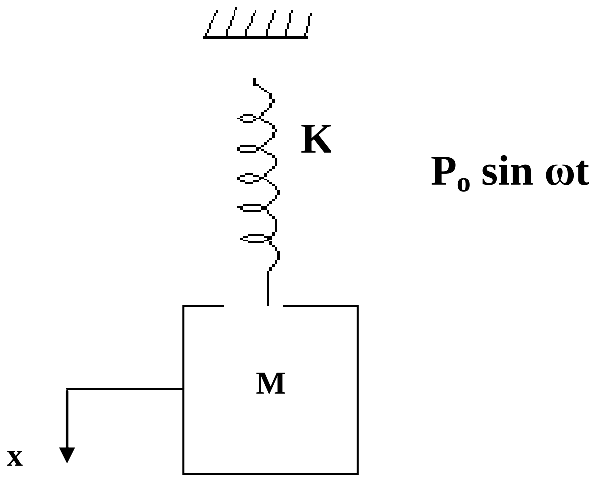
Where k is the spring constant and R is the friction coefficient of the dashpot.

System (b) represents a System undergoing torsional oscillations. It consists of a massive disk of moment of inentia (j) attached to a shaft of torsional stiffiness k. The disk undergoes torsional damping proportional to its angular velocity(θ). The disk has exerted upon it an oscillatory torgue $T_o \sin \omega t$. By Newton's law we have.

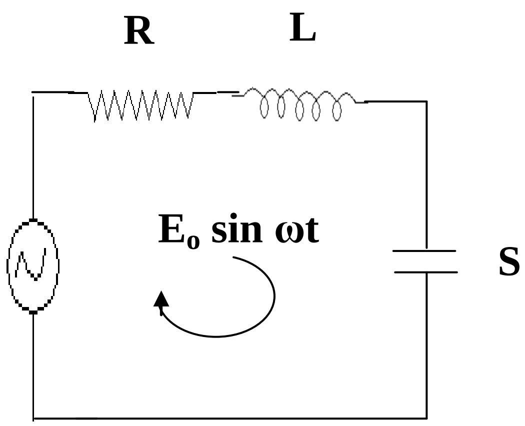
$$J\theta = - k\theta - R\theta + T_o \sin \omega t \qquad (1.1.2)$$

System (c) is a series electrical circuit having inductance , resistance , and elastance. By Kirchhoff's law the equation satisfied by the mesh charge q is

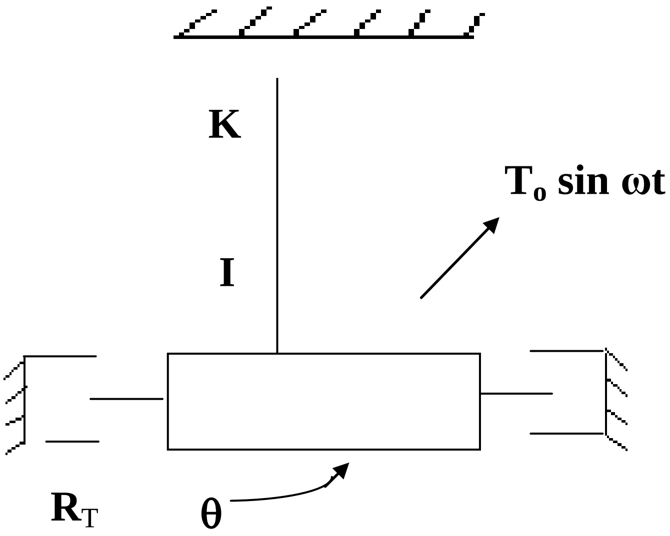
$$L\ddot{q} + R\dot{q} + Sq = E_o \sin\omega t \qquad (1.1.3)$$



(a)



(c)



(b)

Figure (1)

By comparing these three equations , we obtain the following table of analogues.

Linear		**Torsional**		**Electrical**	
Mass	M	Moment of inertia	J	Inductance	L
Sliffiness	K	Torsional stiffness	K	Elastance	S=1/c
Damping	R	Torsional stiffness	R	Resistance	R
Impressed force	$F_o \sin\omega t$	Impressed torgue	$T_o \sin\omega t$	Impressed potential	$E_o \sin\omega t$
Displacement	x	Angular displacement	θ	Capacitor charge	q
Velocity	$\dot{x} = v$	Angular velocity differential Equation	$\dot{\theta} = \omega$	Current	$i = \dot{q}$
$M\ddot{x}+R\dot{x}+kx=p_o \sin\omega t$		$J\ddot{\theta}+R\dot{\theta}+k\theta= T_o \sin\omega t$		$L\ddot{q}+R\dot{q}+sq = E_o \sin\omega t$	

We see from this table of analogues that it is necessary only for us to analzge one system and then by means of the table we may obtain the corresponding solution for others [3] .

1.2. Vibrations in mechanical

Generally speaking vibrations occur whenever a physical system in stable equilibrium is disturbed for then it is subject to forces tending to restore its equilibrium. In the present section we shall see how situations of this kind can lead to differential equations of the form.

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = R(t)$$

and also how the study of these equations sheds light on the physical circumstances.

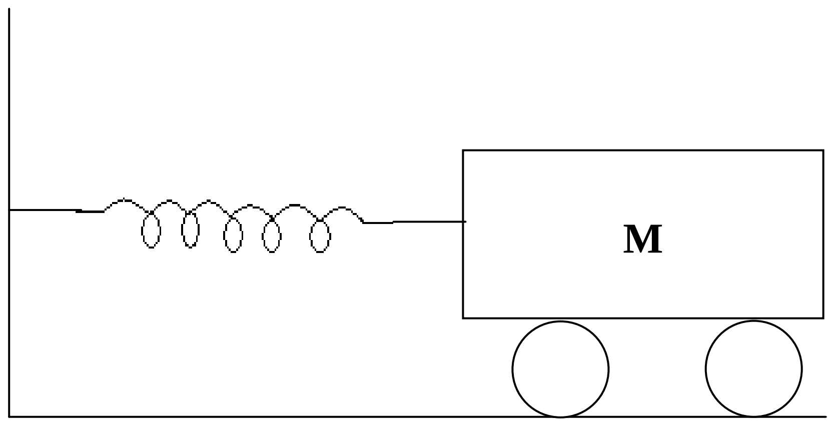


Figure (2)

Undamed simple harmonic vibrations:

As a continuing example, we consider a cart of mass M attached to a nearby wall by means of a spring Figure.1.2.1. The spring exerts no force when the cart is at its equilibrium position x = 0 If the cart is displaced by a distance x, then the spring exerts a restoring force. $F_s = -kx$, where k is a positive constant whose magnitude is a measure of the stiffness of the spring. By Newton's second law of motion, which says that the mass of the cart times its acceleration equals the total force acting on it, we have.

$$M\frac{d^2x}{dt^2} = F_s \qquad (1.2.1)$$

or

$$\frac{d^2x}{} + \frac{k}{}x = o\,, \qquad (1.2.2)$$

dt^2 M

It will be convenient to write this equation of motions in the form

$$\frac{d^2x}{dt^2} + a^2x = o, \qquad (1.2.3)$$

where $a = \sqrt{k/M}$, and its general solution can be written down at once,

$$x = c_1 \sin at + c_2 \cos at. \qquad (1.2.4)$$

If the cart is pulled aside to the position $x = x_o$ and released without any initial velocity at time t = o, so that our initial conditions are

$$x = x_o \text{ and } \upsilon = \frac{dx}{dt} = o \text{ when } t = o, \qquad (1.2.5)$$

Then it is easily seen $c_1 = o$ and $c_2 = x_o$ so(1.2.4) becomes.

$$x = x_o \cos at. \qquad (1.2.6)$$

The amplitude of this simple harmonic vibration is x_o and since its period T is the time required for one complet

e cycle , we have $aT = 2\pi$ and

$$T = \frac{2\pi}{a} = 2\pi\sqrt{\frac{M}{k}} \qquad (1.2.7)$$

Its frequency f is the number of cycles per unit time, so $fT = 1$ and

$$f = \frac{1}{T} = \frac{a}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{M}} \qquad (1.2.8)$$

It is clear from (1.2.8) that the frequency of this vibration increases if the stiffness of the spring is increased or if the mass of the cart is decreased, as our common sense would have led us to predict[1, 2 , 4].

[1] Example (1.2.1):

Consider a spring – mass system with mass of 1 unit and spring constant of 9 units. Suppose the mass is raised 1 units and released with adown ward velocity of 1.2 units. Construct an initial – value problem for this situation , solve it , and describe the properties of the solution.

Example 1.2.1 is an initial – value problem for the linear oscillator equation.

$\ddot{y} + 9y = o$, $\quad y(o) = -0.3$, $\quad \dot{y}(0) = 1.2$

The general solution.

$y = c_1 \cos 3t + c_2 \sin 3t.$

By apply the initial condition we get the solution.

$y = - 0.3 \cos 3t + 0.4 \sin 3t.$

The graph of the solution appears in figure (3) from the graph, we can see that the motion is periodic.

$$T = \frac{2\pi}{a} = \frac{2\pi}{3}$$

$$f = \frac{1}{T} = \frac{3}{2\pi}$$

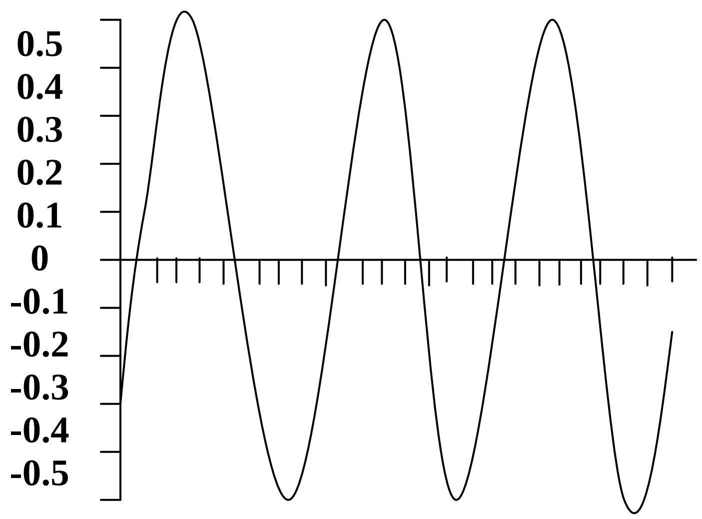


Figure (3)

The solution of $\ddot{y} + 9y = 0, y(0) = -0.3, \dot{y}(0) = 1.2$.

Driven harmonic oscillator:

A driven harmonic oscillator satisfies the non homogenous second order linear differential equation.

$$\frac{d^2x}{dt^2} + a^2x = A_o \cos \omega t,$$

where A_o is the driving amplitude and is the driving frequency for a sinusoidal diving mechanism. This type of system appears in AC LC (inductor – capacitor) circuits and idealized spring systems lacking internal resistance or external air resistance[2] .

Damped Vibrations:

As our next step in developing this physical problem. We consider the additional effect of a damping force F_d due to the viscosity of the medium through which the cart moves (air, water , oil , etc.). We make the specific assumption that this force opposes the motion and has magnitude proportional to the velocity , that is , that F_d = -c (dx/dt), where c is a positive constant measuring the resistance of the medium. Equation (1.2.1) now becomes.

$$M\frac{d^2x}{dt^2} = F_s + F_d \qquad (1.2.9)$$

So

$$\frac{d^2x}{dt^2} + \frac{C}{M}\frac{dx}{dt} + \frac{k}{M}x = o \qquad (1.2.10)$$

Again for the sake of convenience, we write this in the form.

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + a^2x = o. \qquad (1.2.11)$$

where b = c/2M and a= $\sqrt{k/M}$. The auxiliary equation is

$$m^2 + 2bm + a^2 = o \qquad (1.2.12)$$

and its roots m_1 and m_2 are given by (1.2.13)

$$m_1 , m_2 = \frac{-2b \pm \sqrt{4b^2 - 4a^2}}{2} = - b \pm \sqrt{b^2 - a^2}$$

The general solution of (1.2.11) is of course determined by the nature of the numbers m_1 and m_2. As we know , there are three cases , which we consider separately.

CASE A, $b^2 - a^2 > o$ or b > a. In loose terms this amounts to assuming that the frictional force due to the viscosity is large compared ton the stiffness of the spring. It follows that m_1 and m_2 are distinct negative numbers , and the general solution of (11) is

$$x = c_1 e^{m_1 t} + c_2 e^{m_2 t} \qquad (1.2.14)$$

If we apply the initial conditions (1.2.5) to evaluate c_1 and c_2 , (1.2.14) becomes.

$$x = \frac{xo}{m_1 - m_2} (m_1 e^{m2t} - m_2 e^{m2t}). \qquad (1.2.15)$$

The graph of this function is given in Figure (4).

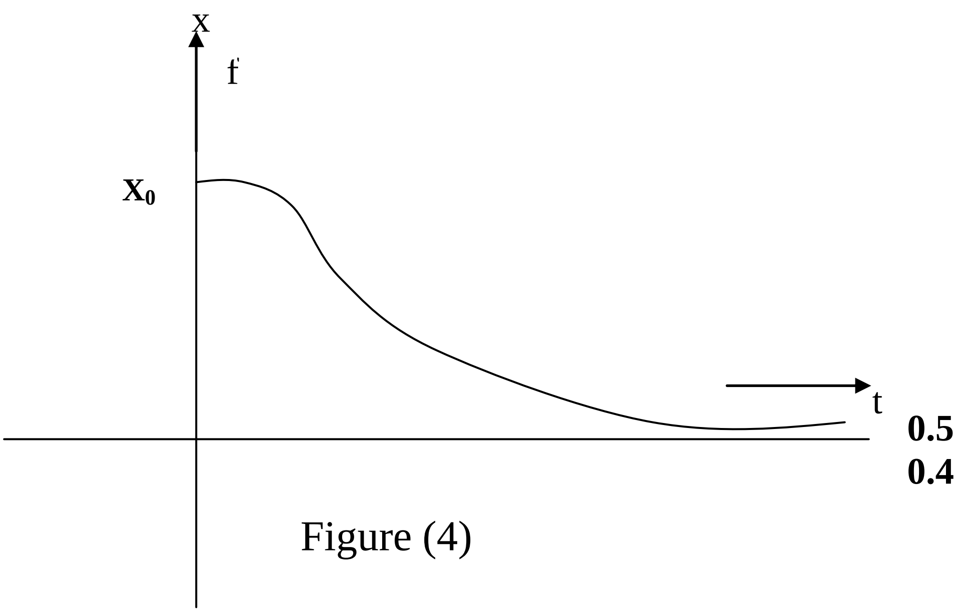


Figure (4)

It is clear that no vibration occurs, and that the cart merely subsides to its equilibrium position. This type of motion is called overdamped. We now imgine that the viscosity is decreased until we reach the condition of next case [1 , 2, 3 , 4].

CASE B. $b^2 - a^2 = o$ or $b = a$ Here we have $m_1 = m_2 = -b = -a$. and the general solution of (1.2.11) is

$$x = C_e^{-at} + C_2 te^{-at} \qquad (1.2.16)$$

When the initial conditions (1.2.5) are imposed, we obtain.

$x = x_o e^{-at} (1 + at)$. (1.2.17)

This function has a graph similar to that of (1.2.15) , and again we have no vibration. Any motion of this kind is said to be critically damped. If the viscosity is now decreased by any amount, however small, then the motion becomes vibratory, and is called underamped. This is the really interesting situation, which we discuss as follows.

CASE C. $b^2 - a^2 < o$ or $b < a$. Here m_1 and m_2 are con jugate complex numbers $- b \pm c\, \alpha$, where

$\alpha = \sqrt{a^2 - b^2}$,

And the general solution of (1.2.11) is

$x = e^{-bt} (c_1 \cos \alpha t + c_2 \sin \alpha t)$. (1.2.18)

When c_1 and c_2 are evaluated in accordance with the invtial conditions (1.2.5) , this becomes.

$$x = \frac{x_0}{\alpha} e^{-bt} (\alpha \cos \alpha t + b \sin \alpha t). \qquad (1.2.19)$$

If we introduce $\theta = \tan^{-1} (b/\alpha)$, then (1.2.19) can be expressed in the more revealing form.

$$x = \frac{x_0 \sqrt{\alpha^2 + b^2}}{\alpha} e^{-bt} \cos(\alpha t - o). \qquad (1.2.20)$$

This function oscillates with amplitude that falls off exponentially, as Figure. (5) Shows. It is not periodic in the strict sense , but its graph crosses the equilibrium position x= o at regular intervals. If we consider its "period" T as the time required for one complete "cycle".

Then $\alpha T = 2\pi$ and

$$\boxed{T =} \frac{2\pi}{\alpha} = \frac{2\pi}{\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{k/m - C^2/4M^2}}$$

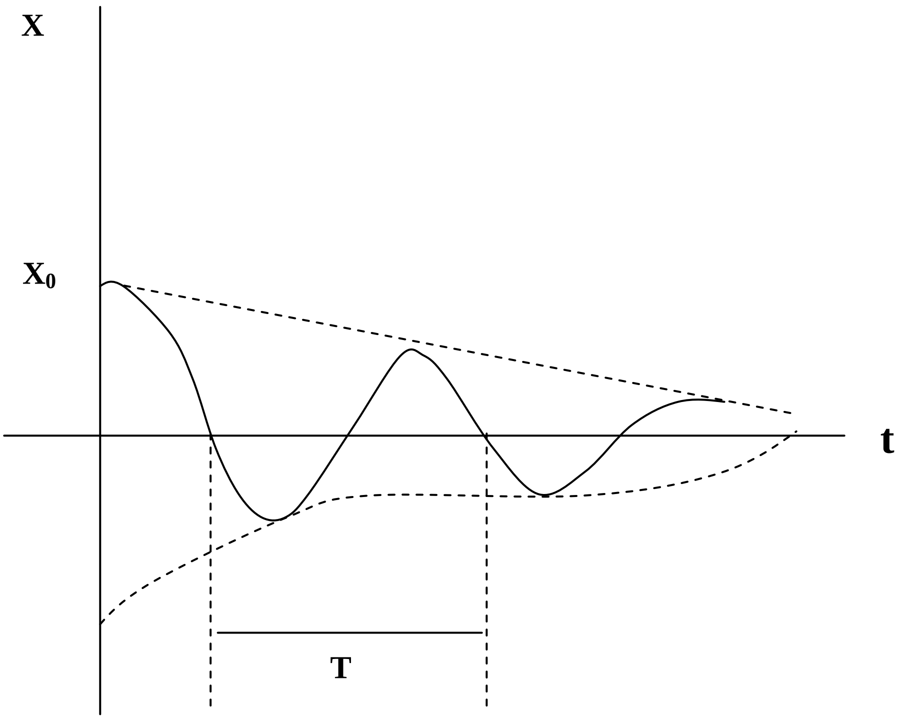


Figure (5)

Also , its " frequency" f is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{a^2 - b^2} = \frac{1}{2\pi}\sqrt{k/M - c^2/4M^2}$$

This number is usually called the natural frequency of the system. When the viscosity vanishes, so that c = O , it is clear that (1.2.21) and (1.2.22) reduce to (1.2.7) and (1.2.8.8). Furthermore, on comparing (1.2.8) and (1.2.22) we see that the frequency of the vibration is decreased by damping, as we might expect. (1.1,2).

Forced vibrations: The vibrations discussed above are known as free vibrations because all the forces acting on the system are internal to the system itself. We now extend our analysis to cover the case in which an impressed external force Fe = f (t) acts on the cart. Such a force might arise in many ways , for vibrations of the wall to which the spring is attached , or from the effect on the cart of an external magnetic field (if the cart is made of iron). In place of (9) we now have.

$$M \frac{d^2x}{dt^2} = F_s + F_d + F_e \qquad (1.2.23)$$

Or

$$M + \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = f(t). \qquad (1.2.24)$$

The most important case is that in which the impressed force is periodic and has the form of f (x) = F_o cosωt, so that (1.2.24) becomes.

$$M \frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_o \cos\omega t. \qquad (1.2.25)$$

We have already solved the corresponding homogeneous equation (1.2.10), so in seeking the general solution of (1.2.25) all that remains is to find a particular solution. This is most readily accomplished by the method of undetermined coefficients. Accordingly, we take $x = A \sin \omega t + B \cos\omega t$ as a trial solution. On substituting this into (1.2.25) we obtain the following pair of equations for A and B,

$$\omega c A + (k - \omega 2M) B = F_o ,$$

$$(k - \omega 2M) A - wcB = o.$$

The solution of this system is

$$A = \frac{\omega c F_o}{(k-\omega^2 M)^2+\omega^2 c^2} \quad \text{and} \quad B = \frac{(k- \omega^2 M)F_o}{(k-\omega^2 M)^2+\omega^2 c^2}$$

Our desired particular solution is therefore

$$x = \frac{F_o}{(k-\omega^2 M)^2+\omega^2 c^2}\left[\omega c \sin \omega t + (k-\omega^2 M) \cos \omega t\right] \qquad (1.2.26)$$

By introducing $\varphi = \tan^{-1} (\omega c/k-\omega 2M)$, we can write (1.2.26) in the more useful form.

$$x = \frac{F_o}{\sqrt{(k-\omega^2 M)^2+w^2 c^2}} \cos(\omega t - \varphi) \qquad (1.2.27)$$

If we now assume that we are dealing with the under damped motion discussed above, then the general solution of (1.2.25) is

$$x = e^{-bt}(c1 \cos \alpha t + c2 \sin \alpha t) - \frac{Fo}{\sqrt{(k - \omega^2 M)^2+\omega^2 c^2}} \cos(\omega t-y)$$

The first term here is clearly transient is the sense that it approaches O as $t \longrightarrow \infty$ .As a matter of fact , this is true whether the motion is under damped or not , as long as some degree of damping is present. Therefore, as time goes on, the motion assumes the character of the second term, the steady. state part. On this basis, we can neglect the transient part of (1.2.28) and assert that for large the general solution of (1.2.25) is essentially equal to the particular solution (1.2.27).The frequency of this forced vibration equals the impressed frequency $\omega/2\pi$, and its am plitude is the coefficient

$$\frac{F_o}{\sqrt{(k-\omega^2 M)^2+\omega^2 c^2}} .$$

This expression for the amplitude holds some interesting secrets , for it depends not only on ω and F_o but also on k , c , and M. As an example , we note that if c is very small and ω is close to $\sqrt{k/m}$ (so that $k - \omega^2 M$ is very small) , which means that the motion is lightly damped and the impressed frequency $\omega/2\pi$ is close to the natural frequency.

$$\frac{1}{2\pi}\sqrt{\frac{k}{M} - \frac{c^2}{4M^2}} ,$$

then the amplitude is very large. This phenomenon is known as resonance. A classic example is provided by the forced vibration of a bridge under the impact of the feet of marching columns of men whose pace corresponds closely to the natural frequency of the bridge [1,2,3,4] .

1.3 Discharge and charge of a capacitor

An interesting application of the differential equation governing the distribution of charges and currents in electrical networks is the following one.

Consider the electrical circuit of Figure (6) .Let a charge q_o be placed on the capacitor, and let the switch S be closed at t = 0 . Let it be required to determine the charge on the capacitor at any instant later.

When the switch is closed, we have, by Kirchhoff's law, the equation.

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{c} = 0 \qquad (1.3.1)$$

To solve this , let us introduce the transform

$$\ell q = Q \qquad (1.3.2)$$

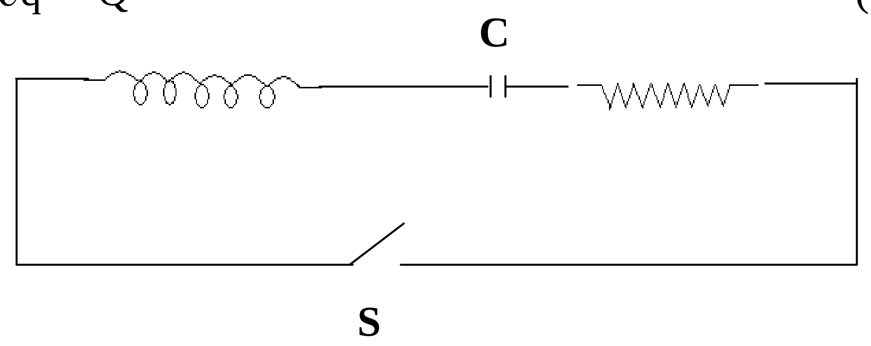


Figure.(6)

The initial conditions of the problem are

$$\left.\begin{array}{l} q = q_o \\ i = \dfrac{dq}{dt} = 0 \end{array}\right\} \text{ at } t = 0 \qquad (1.3.3)$$

Hence we have

$$\ell \; \frac{dq}{dt} = sQ - q_o \qquad (1.3.4)$$

$$\ell \; \frac{d^2q}{dt^2} = s^2Q - sq_o$$

Hence Equation. (1.3.1) transforms to

$$L\,(s^2Q - sq_o) + R\,(s\,Q - q_o) + \frac{Q}{C} = o \qquad (1.35)$$

Or

$$(S^2\,L + sR + 1/c)\,Q = Ls\,q_o + Rq_o \qquad (1.3.6)$$

Let

$$a = \frac{R}{2L} \qquad \omega_o = \sqrt{\frac{1}{Lc}} \qquad (1.3.7)$$

We , therefore have

$$Q = \frac{sq_o}{s^2 + 2\,as + w_o^2} + \frac{2a\,q_o}{s^2 + 2as + \omega_o^2} \qquad (1.3.8)$$

By the use of transforms Noumbers , 2.22 and 2.23 of the table of Laplace transforms {3} , we obtain .

$$q = q_o\,e^{-at}\,(\cosh ßt + \frac{a}{ß}\sinh ßt) \text{ if } a > \omega_o$$

$$q = q_o\,e^{-at}\,(1 + at) \text{ if } a = \omega_o$$

$$q = q_o\,e^{-at}\,(\cos \omega st + \frac{a}{\omega_s}\sin \omega st) \text{ if } a < \omega_o \qquad (1.3.9)$$

$$q = q_o\,e^{-at}\,(\cos \omega st \; \frac{a}{\omega_s}\sin \omega st) \text{ if } a < \omega_o$$

Where $\omega_s = \sqrt{\omega_o^2 - a^2}$ and $B = \sqrt{a^2 - \omega_o^2}$

The charging of A capacitor:

Let us consider the circuit of Figure (7). In this case, at t = o, the switch is closed and the potential E of the battery is impressed on the circuit, It is required to determine the manner in which the charge on the capacitor behaves. The equation satisfied by the charge is now

$$\ddot{q} + 2aq + \omega o^2 \dot{q} = \frac{E}{L} \qquad (1.3.10)$$

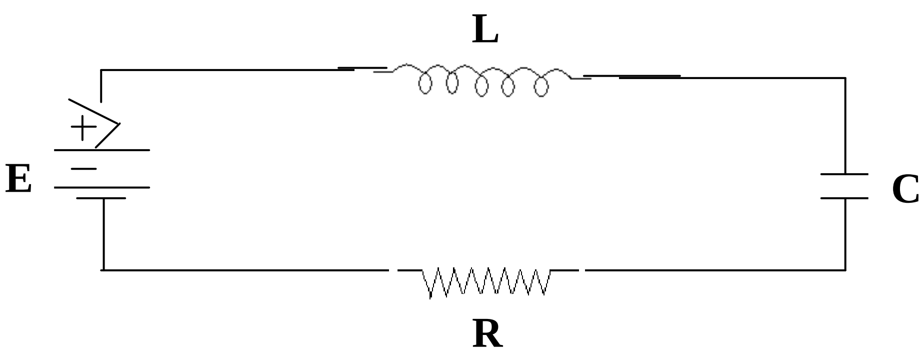


Figure. (7)

To solve this equation , we again let

$$\ell q = Q \qquad (1.3.11)$$

And since E is a constant, we have

$$\ell \frac{E}{L} = \frac{E}{Ls} \qquad (1.3.12)$$

The initial conditions are now

$$\left.\begin{array}{l} q = o \\ q = o \end{array}\right\} \text{at } t = o \qquad (1.3.13)$$

Hence we have

$$\ell \ddot{q} = s^2 Q \qquad (1.3.14)$$

.

$\ell q = sQ$

Equation (1.3.10) transforms to

$$(s^2 + 2as + \omega_o^2)\, Q \frac{E}{Ls} \qquad (1.3.15)$$

And hence

$$Q = \frac{E}{L(s^2 + 2as + \omega_o^2)} \qquad (1.3.16)$$

To obtain the inverse transform of (1.3.16) , we must use transform No 3.32 of the table of Laplace transforms[3] and this obtain .

$$\left.\begin{aligned} q &= CE\left(1 - e^{-at}\left(\cosh \beta t + \frac{a}{B} \sinh \beta t\right)\right) && \text{if } a > \omega_o \\ q &= CE\left(1 - e^{-at} (1 + at)\right) && \text{if } a = \omega_o \\ q &= CE\left(1 - e^{-at} \left(\cos \omega st + \frac{a}{\omega s} \sin \omega st\right)\right) && \text{if } a < \omega_o \end{aligned}\right\} (1.3.17)$$

Where β and ωs are as defined in (9).

In each case , the charging current is given by

i = q. The analogous mechanized problem is that of determining the motion of a mass when it has been given an initial displacement and is acted upon by a spring and retarded by viscous friction or if the mass has a sudden force applied to it[3].

1.4 A Model for flow of water in a pump , Tank , and pipe system:

Figure (8) illstrates a device consisting of a pump (E), a storage tank (C) , a pair of narrow rigid pipes (Ri) , and a switch (s) that can open either of the pipes. The pump pushes water up pipe R_1 with a constant amount of pressure. If that pipe is open, water rises through it into the storage tank. Similarly, if the switch is set so that pipe R_2 is open, then water drains from the tank under the force of its own weight. Both pipes can be closed, but the switch does not permit both pipes to be open at the same time.

Given this simple. Conceptual model, we can use basic principles of water flow to obtain a mathematical model. Let V(t) be the volume of water in the tank. Let i(t) be the flow rate of the water through either open pipe. The flow rate is the rate of change of the volume, so $\dot{V} = i$. Let E(t) be the pressure produced by the pump, let $V_c(t)$ be the pressure of the water in the tank and let $V_R(t)$ be the pressure associated with the movement of water in whichever pipe is open. We have to determine the connection among the three pressures, and we have to connect the pressures V_c and V_R with the volume of the tank or the flow rate through the pipe. We begin with the relationship of the pressures. The pressure in an open pipe results from the combination of the pressure from below and the pressure from above. V_E be the pressure from below ($V_E = E$ when pipe R_1 is open and $V_E = o$ when pipe R_2 is open). Then the net pressure from below is the pressure from below minus the pressure from above, using the notation of the model , we have $V_R = V_E - Vc$, or.

$$V_R + Vc = V_E \qquad (1.4.1)$$

The pressure of the water in the water in the tank is proportional to the height of the water column, which is simply the volume divided by the constant cross – sectional area.

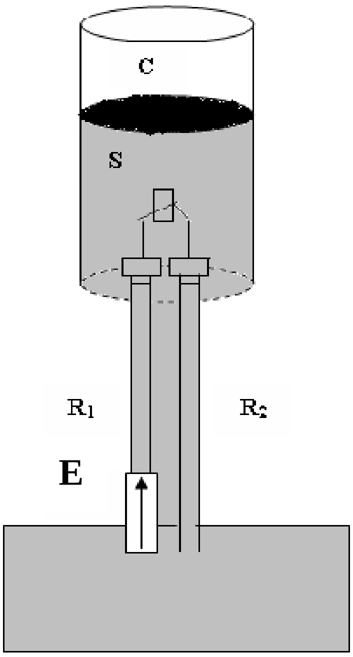


Figure (8): Aconepetual model for a water flow problem

We have

$$Vc = \frac{V}{C}, \qquad (1.4.2)$$

Where C is the cross – sectional area multiplied by , a proportionality constant. Given two tanks of different cross – sectional, each containing the same volume, the one with. The smaller cross – sectional area will produce a greater pressure because the water column will rise higher. The flow of water in either pipe is caused by pressure in the pipe. In particular the flow rate should be proportional to the pressure. Let R be a constant that measures the resistance of the pipe to flow. Then $i = \frac{V_R}{R}$

Or

$$V_R = R\,i = R\dot{V} \qquad (1.4.3)$$

We can combine Equation (1) and (2) to obtain a formula for V_R in terms of V_c,

$$V_R = R\dot{V} = R_C\,\dot{V}_c.$$

Substituting this result into the pressure balance equation (1) yields the differential equation.

$$RC\,\dot{V}_c + V_c = V_E.$$

At this point, the notation can be simplified by dropping the subscript from V_c. From here on, V without a subscript indicates V_c. In its final form, we have the model.

$$RC\,\dot{V} + V = V_E \qquad (1.4.4)$$

The conceptual model for the flow of water can also be thought of as a conceptual model for an RC series circuit, as depicted in Figure (9). The water volume V corresponds to the electric charge q. The pipe and tank correspond to circuit elements called a resistor and capacitor , respectively, the quantities Vc, V_R , E, and i represent the voltage measured across the compactor , the voltage measured across the resistor, the voltage (or electromotive force) produced by the power supply , and the electric current. The properties R and C are the resistance and capacitance , respectively , of the electric circuit , measured in ohms (Ω) and farads (F). Voltage and current are measured in volts (V) and amperes(A).

Whatever results we obtain for the differential equation (4) can be interpreted in the language of electric circuits or the language of water flow.

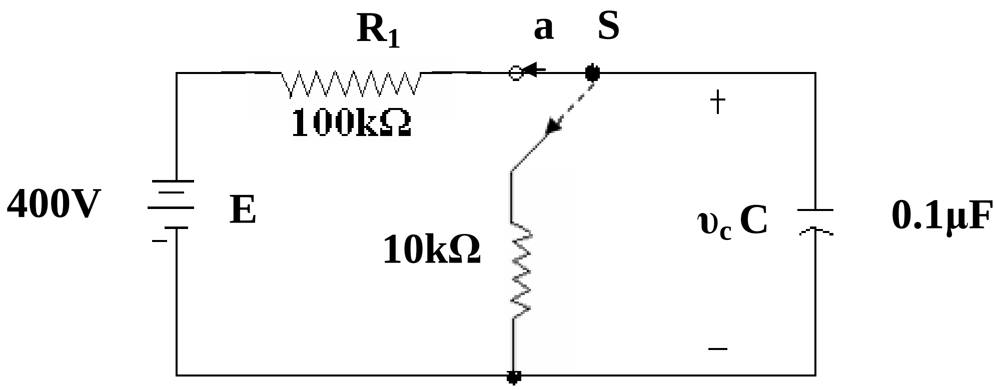


Figure (9)

Example 1.4.1

Consider the water flow scenario of Figure (8) and the equivalent electric circuit scenario of Figure (9). Suppose the switch is initially set so that the tank is not open to either pipe , and there is initially no water in the tank. At time o , the switch is set so that pipe R_1 is open , thus , V_E = E AT t = 0.03, the switch is reset so that pipe R_2 is open , corresponding to V_E = o.we want to determine the behavior of the system , as given by v (t) and i (t).

The initial – value problem for the tank pressure before t = 0.03 is.

$R_1 C\dot{V} + V = E$, V (0) = 0.

This differential equation is a decay equation, so it can be solved by using the translation y = V – E to yield the homogeneous problem.

$R_1 c\dot{y} + y = 0$, y (0) = -E-

This problem has the solution

$y = - E e^{-t/(R_1 c)}$,

$V = E(1 - e^{-t/(R_1 C)})$.

Using the parameter values $R_1 = 10^6$. $C = 10^{-7}$, and E = 400, as given in figure we get.

$V = 400 (1 - e^{-100t})$.

This solution holds as long as the switch is open to the pump. In particular, let V_1 be the pressure at t = 0.03 , then

$V_1 = V (0.03) = 400 (1-e^{-3}) \approx 380$.

At t = 0.03 , the movement of the switch changes the relevant problem. The pressure now satisfies the initial – valiue problem.

$R_2 C\dot{V} + V = 0$, $V(0.03) = V_1$

This problem is homogeneous, and easily solved, with the result.

$V = V_1 e^{-(t-0.03)/(R2c)}$,

Given the parameter values , this is

$V = 400 (1 – e^{-3}) e^{- 1000 t + 30}$

Altogether, we have a complete formula for the pressure ,

$$V = 400 \begin{cases} 1- e^{-100t} & t < 0.03 \\ (1 - e^{-3})\, e^{-1000t+30} & t > 0.03 \end{cases}$$

Differentiating this formula give the result

$$i = C\dot{V} = 0.04 \begin{cases} 0.1e^{-100t} & t < 0.03 \\ -(1-e^{-3})\ e^{-1000t+30} & t > 0.o3 \end{cases}$$

For the flow rate.

The solution is illustrated in Figures (10) and (11).

The tank stores water , and since the pressure is proportional to the amount of water , we can think of the tank as storing pressure to be converted to flow at some later time. In the two stages of the problem, the tank is first charged and then drained, with each stage accomplished through the flow of water. The resistance to flow is less in the second pipe than in the first, hence, the draining process is faster, and the flow rate during draining is larger in magnitude. Similarly, the capacitor in the electric circuit stores charge and voltage, we can think of the stored voltage as a potential foe future current. The capacitor is charged and then discharged, with the rates of these processes, and the associated currents, dependent on the relative resistances of the associated portion of the circuit [1] .

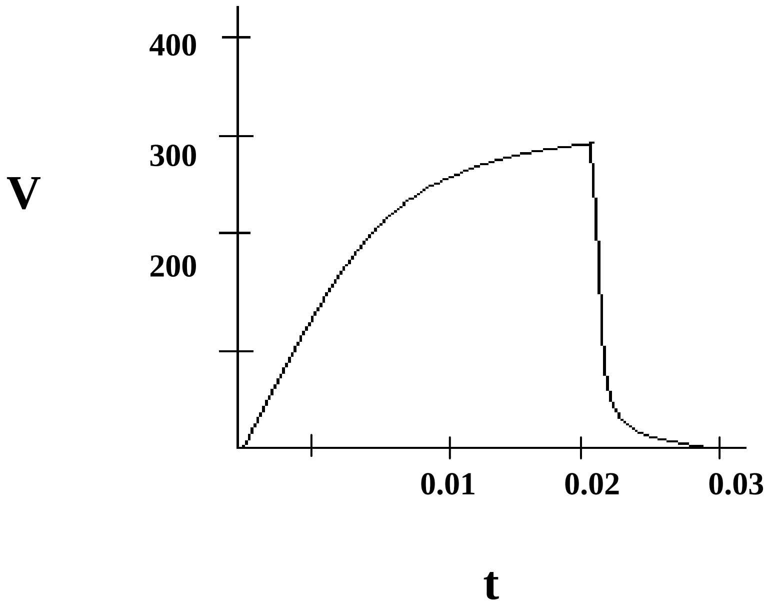


Figure (10)

The tank pressure for the flow problem of Example 1.

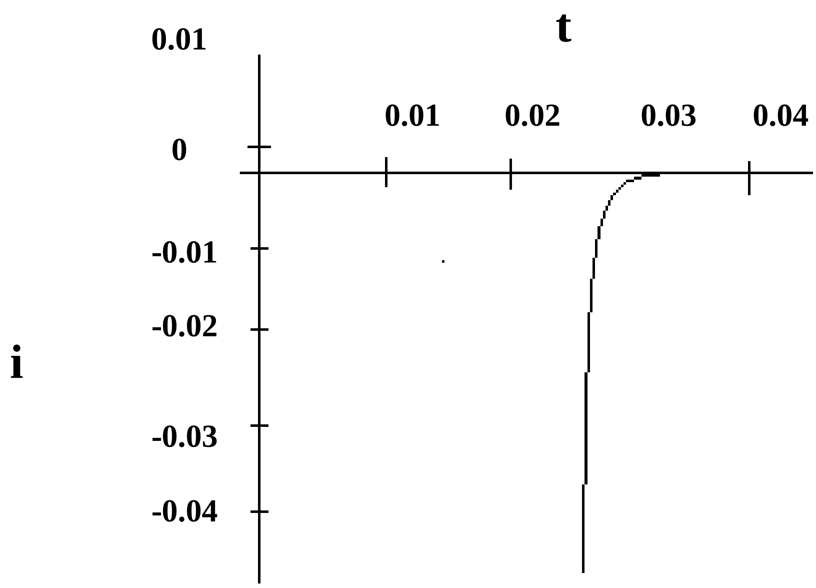


Figure (11)

The flow rate for the flow problem of Example 1.

1.5 Circuit with mutual in ductance:

Let us consider the circuits of Figure (12) . In this case we have two circuits coupled magnetically. The coefficient L_{12} is termed the mutual inductance coefficient. It is positive if the magnetic fields of i_1, and i_2 is add. If they are opposed, then the coefficient L_{12} is negative. In any case the equation governing the currents in the two circuit are given by applying Kirchhoff's laws to the two loops and are.

$$\begin{aligned} L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} + R_{11} i_1 &= E \\ L_{22}\frac{di_2}{dt} + L_{12}\frac{di_1}{dt} + R_{22} i_2 &= o \end{aligned} \qquad (1.5.1)$$

We wish to determine the currents i_1 and i_2 on the supposition that at $t = 0$ the switch s is closed and the initial currents are zero. Let us introduce the transforms

$$\begin{aligned} \ell\, i_1 &= I_1 \\ \ell\, i_2 &= I_2 \end{aligned} \qquad (1.5.2)$$

Now since we have

$$\left.\begin{aligned} i_1 &= o \\ i_2 &= o \end{aligned}\right\} \text{ at } t = o \qquad (1.5.3)$$

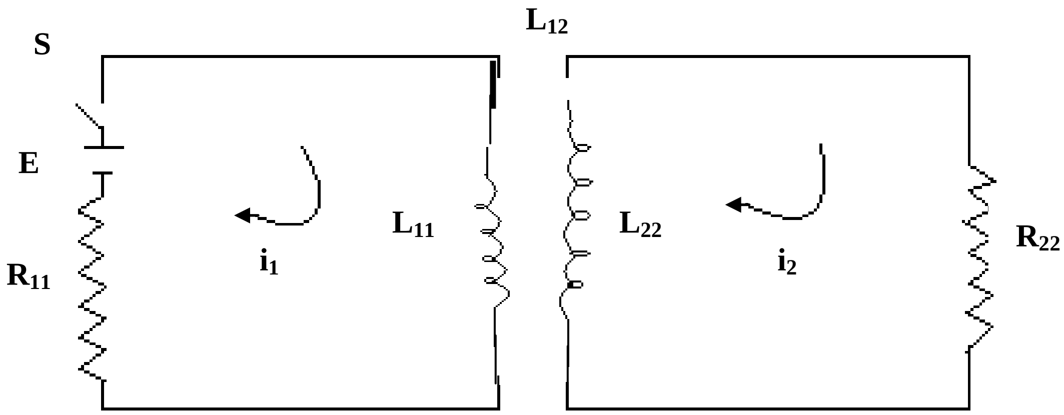
and also E is a constant , Equation.(1.5.1) transforms to

$$sL_{11}\ I_1 + s\ L_{12}\ I_2 + R_{11}\ I_1 = \frac{E}{s}$$
$$s\ L_{22}\ I_2 + s\ L_{12}\ I_1 + R_{22}\ I_2 = o \qquad (1.5.4)$$

We now solve these two algebraic equations by using crammer's rule and obtain

$$I_1 = \frac{\begin{vmatrix} \frac{E}{s} & sL_{12} \\ O & sL_{22} + R_{22} \end{vmatrix}}{\begin{vmatrix} sL_{11} + R_{11} & sL_{12} \\ sL_{12} & sL_{22}+R_{22} \end{vmatrix}} \qquad (1.5.5)$$

$$I_2 = \frac{\begin{vmatrix} sL_{11}+R_{11} & \frac{E}{s} \\ sL_{12} & O \end{vmatrix}}{\begin{vmatrix} sL_{11} + R_{11} & sL_{12} \\ sL_{12} & sL_{22}+R_{22} \end{vmatrix}}$$



Figure(12)

Hence we have

$$I_1 = \frac{E(sL_{22} + R_{22})}{\left[(L_{11}\,L_{22} - L^2_{12})\,s^2 + (R_{11}\,L_{22} + R_{22}\,L_{11})s + R_{11}R_{22}\right] s} \qquad (1.5.6)$$

If we let

$$a = \frac{R_{11}\,L_{22} + R_{22}\,L_{11}}{2(L_{11}\,L_{22} - L^2_{12})} \qquad (1.5.7)$$

And

$$\omega_o^2 = \frac{R_{11}\,R_{22}}{L_{11}\,L_{22} - L^2_{12}} \qquad (1.5.8)$$

We then have

$$I_1 = \frac{E}{L_{11}\,L_{22} - L^2_{12}} \; \frac{sL_{22} + R_{22}}{(s^2 + 2as + \omega_o^2)s}$$

$$I_2 = \frac{-E}{L_{11}L_{22} - L^2_{12}} \; \frac{L_{12}}{s^2 + 2as + \omega_o^2} \qquad (1.5.9)$$

In this case

$$a^2 > \omega_o^2 \quad \sqrt{a^2 - \omega o^2} = \beta \qquad (1.5.10)$$

Using the transforms nos. 2.23 and 3.23 of the table of transforms[3] , we obtain after some algebraic reductions.

$$i_1 = \frac{E}{R_{11}}\left(1 - e^{-at}\cosh \beta t + \frac{(a^2 - \beta^2)\,L_{22} - aR_{22}}{\beta R_{22}}\,e^{-at}\sinh \beta t\right) \qquad (1.5.11)$$

$$i_2 = \frac{(\beta^2 - a^2)L_{12}E}{\beta\,R_{11}\,R_{22}}\,e^{-at}\sinh \beta t$$

for the transforms of I_1 and I_2. We see that, as time el apse, i_1 approaches its final value E/Ru. If we set $di_2/dt = o$ and solve for t , we find that i_1 rises to a maximum value when.

$$t = \frac{1}{\beta}\tanh^{-1}\frac{\beta}{a} \qquad (1.5.12)$$

and then approaches zero asymptotically. An interesting special case is the symmetrical one. In this case the resistances of each mesh are equal, and the self- inductances are equal. We then have.

$$R_{11} = R_{22} = R \qquad L_{11} = L_{22} = L$$
$$L_{12} = M \qquad (1.5.13)$$

Equations (1.5.4) then become

$$sL\, I_1 + s\, MI_2 + RI_1 = \frac{E}{s}$$
$$sL\, I_2 + sMI_1 + RI_2 = o \qquad (1.5.14)$$

If we add the two equations, we obtain

$$sL\,(I_1 + I_2) + sM\,(I_1 + I_2) + R\,(I_1 + I_2) = \frac{E}{s} \qquad (1.5.15$$

If we subtract the second equation from the first one,we have.

$$sL\,(I_1 - I_2) - sM\,(I_2 - I_2) + R\,(I_1 - I_2) = \frac{E}{s} \qquad (1.5.16)$$

If we now let

$$x_1 = I_1 + I_2 \qquad x_2 = I_2 - I_2 \qquad (1.5.17)$$

We have

$$s(L + M)\,x_1 + Rx_1 = \quad E$$

$$\overline{\quad} \atop s \qquad (1.5.18)$$

$$s(L + M)\, x_1 + Rx_1 = \frac{E}{s}$$

Hence

$$x_1 = \frac{E}{\left[s(L+M) + R \right] s} \qquad x_2 = \frac{E}{\left[s\,(L-M)+R \right] s} \qquad (1.5.19)$$

If we let

$$a_1 = \frac{R}{L+M} \qquad a_2 = \frac{R}{L-M} \qquad (1.5.20)$$

We obtain

$$x_1 = \frac{E}{L+M} \quad \frac{1}{(s+a_1)s} \qquad x_2 = \frac{E}{L-M} \quad \frac{1}{(s+a_2)s} \qquad (1.5.21)$$

Using transform No. 2.1 of the table of Laplace transforms [3] ,we have.

$$\ell^{-1} x_1 = \frac{E}{R}(1-e^{-a_1 t}).\ell^{-1} x_2 = \frac{E}{R}(1-e^{-a_2 t}) \qquad (1.5.22)$$

Hence

$$i_1 + i_2 = \frac{E}{R}(1 - e^{-a_1 t})$$

$$i_1 - i_2 = \frac{E}{R} \quad (1- e^{-a_2} t) \qquad (1.5.23)$$

And adding the two equation we obtain

$$i_1 = \frac{E}{R} \quad \frac{2 - e^{-a_1 t} - e^{-a_2 t}}{2} \qquad (1.5.24)$$

Subtracting the second equation from the fist equation, we obtain.

$$i_1 = \frac{E}{2R}(e^{-a_2t} - e^{-a_1t})$$

These are the currents in the symmetrical case [3].

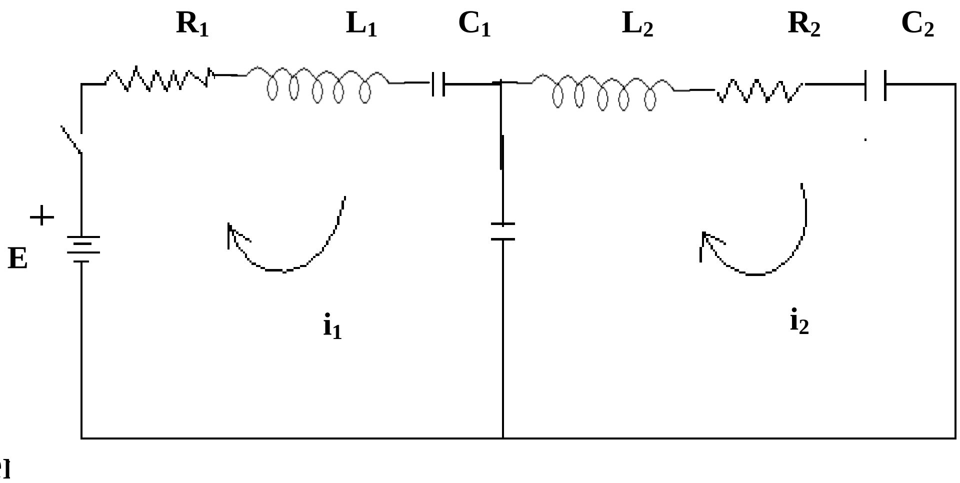
1.6. Circuits coupled by a capacitor:

let us consider the circuit of Figure (13). In this case we have two coupled circuits. The coupling element is now a capacitor. Let the switch S. be closed at t = o, and let it be required to determine the current in the system. We write Kirchhoff's law for both meshes. We then obtain.

$$L_1\frac{di_1}{dt} + R_1i_1 + \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_{12}} = E$$

-

$$L_2\frac{di_2}{dt} + R_2i_2 + \frac{q_2}{C_2} + \frac{q_2 - q_1}{C_{12}} = 0 \qquad (1.6.1)$$



Wher

$$i_1 = \frac{dq_1}{dt} \qquad i_2 = \frac{dq_2}{dt} \qquad (1.6.2)$$

let us introduce the transforms

$$\ell i_1 = I_1 \qquad \ell q_1 = Q_1$$
$$\ell i_2 = I_2 \qquad \ell q_2 = Q_2 \qquad (1.6.3)$$

If we assume

$$\left.\begin{matrix} i_1 = o & q_1 = o \\ i_2 = o & q_2 = o \end{matrix}\right\} \quad \text{at } t = 0 \qquad (1.6.4)$$

Equation (1.6.1) transform into

$$sL_1 I_1 + R_1 I_1 + I_1 \frac{I_1}{sC_1} - \frac{I_1 - I_2}{sC_{12}} = \frac{E}{s}$$
$$(1.6.5)$$
$$sL_2 I_2 + R_2 I_2 + \frac{I_2}{sC_2} + \frac{I_2 - I_1}{sC_{12}} = 0$$

We now solve these equations algebraically for the transforms I_2 and I_2. We this obtain.

$$I_1 = \frac{\begin{vmatrix} \frac{E}{s} & \frac{-1}{sC_{12}} \\ o & sL2 + R2 + \frac{1}{sC_2} + \frac{1}{sC_{12}} \end{vmatrix}}{\begin{vmatrix} sL_1 + R_1 + \frac{1}{sC_1} + \frac{1}{sC_{12}} & -\frac{1}{sC_{12}} \\ -\frac{1}{sC_{12}} & sL_2 + R_2 + \frac{1}{sC_2} + \frac{1}{sC_{12}} \end{vmatrix}} \qquad (1.6.6)$$

$$I_2 = \frac{\begin{vmatrix} sL_1 + R_2 + \frac{1}{sC_1} + \frac{1}{1C_{12}} & \frac{E}{s} \\ -\frac{1}{sC_{12}} & O \end{vmatrix}}{\begin{vmatrix} sL_1 + R_1 + \frac{1}{sC_1} + \frac{1}{sC_{12}} & -\frac{1}{sC_{12}} \\ & \end{vmatrix}}$$

$$- \frac{1}{sC_{12}} \qquad sL_2 + R_2 + \frac{1}{sC_2} + \frac{1}{sC_{12}}$$

Expanding the determinants, we obtain the transforms I_1, and I_2 as the ratios of polynomials in s. the inverse transforms of I, and I_2 give the currents in the system. In this case , the determinant in the denominator of (1.6.6) of the system is a polynomial of the fourth degree in s No. T. 17 in the table of Laplace transforms[3]. This entails the solution of a quadratic equation in s. the trend of the general solution may be determined by solving the symmetrical case in which we have.

$$R_1 = R_2 = R \qquad C_1 = C_2 = C \qquad L_1 = L_2 = L \qquad (1.6.7)$$

In this case Equation (1.6.5) reduce to

$$sL\, I_1 + RI_1 + \frac{I_1}{sC} + \frac{I_1}{sC_{12}} - \frac{I_2}{sC_{12}} = \frac{E}{s} \qquad (1.6.8)$$

$$sL\, I_2 + RI_2 + \frac{I_2}{sC} + \frac{I_2}{sC_{12}} - \frac{I_1}{sC_{12}} = 0$$

If we add the two equations, we obtain

$$sL\,(I_1 + I_2) + R\,(I_1 + I_2) + \frac{I}{sC}(I_1 + I_2) = \frac{E}{s} \qquad (1.6.9)$$

If we subtract the second equation from the first, we have.

$$sL\,(I_1 - I_2) + R\,(I_1 + I_2) + \frac{1}{sC}(I_1 - I_2)$$

$$+\frac{1}{sC_{12}}(I_1 - I_2) + \frac{I}{sC_{12}}(I_1 - I_2) = \frac{E}{s} \qquad (1.6.10$$

If we let

$$x_1 = I_1 + I_2$$
$$x_2 = I_1 + I_2 \qquad (1.6.11)$$

We obtain

$$sLx_1 + Rx_1 + \frac{1}{sC}x_1 = \frac{E}{s}$$
$$sLx_1 + Rx_2 + \frac{1}{s}\left(\frac{1}{C} + \frac{2}{C_{12}}\right)x_2 = \frac{E}{s} \qquad (1.6.12)$$

If we let

$$\frac{R}{2L} = a \qquad \omega_1^2 = \frac{1}{LC} \qquad \omega_2^2 = \frac{1}{L}\left(\frac{1}{C} + \frac{2}{C_{12}}\right) \qquad (1.6.13)$$

The two equations become

$$(s^2 + 2as + \omega_1^2)\, x_1 = \frac{E}{L}$$
$$(s^2 + 2as + \omega_2^2)\, x_2 = \frac{E}{L} \qquad (1.6.14)$$

The inverse transforms of x_1 may now be calculated by No. 2.22 in the table of transforms(3) .In this the case.

$$\omega_1^2 > a^2 \qquad \omega_2^2 > a^2 \qquad (1.6.15)$$

We have

$$\ell^{-1}\, x_1 = \frac{E}{L\omega_a}(e^{-at} \sin \omega_a t) \qquad \omega_a = \sqrt{\omega_1^2 - a^2} \qquad (1.6.16)$$

$$\ell^{-1}\ x_2 = \frac{E}{L\omega_b}(e^{-at}\sin\omega_b t) \qquad \omega_b = \sqrt{\omega_1^2 - a^2}$$

Hence , adding the two equations (1.6.16) , we have

$$i_1 = \frac{E}{2L}e^{-at}\left(\frac{\sin\omega_a t}{\omega_a} + \frac{\sin\omega_b t}{\omega_b}\right)$$

$$i_1 = \frac{E}{2L}e^{-at}\left(\frac{\sin\omega_a t}{\omega_a} + \frac{\sin\omega_b t}{\omega_b}\right) \qquad (1.6.17)$$

If there is no resistance in the circuit, then a = o and the currents oscillate without loss of amplitude with the angular frequencies ω_1 and ω_2 [3] .

----------------------------------Chapter Tow

Oscillation of non linear systems

2-1 An operational analysis of nonlinear dynamical systems:

A powerful method of determining the free oscillations of certain nonlinear systems will be given in this section. The method presented here is an operational adaptation of the one developed by Linstedt and Liapounoff .

The method may be illustrated by a consideration of a mechanical oscillating system consisting of a mass attached to a spring. The equation of the free vibration of such a system is

$$\frac{md^2x}{dt^2} + F(x) = 0 \qquad (2.1.1)$$

where md^2x/dt^2 is the inertia force of the mass, F(x) is the spring force, and x is measured from the position of equilibrium of themass when the spring is not stressed. Let us consider the symmetrical case where.

$$F(x) = kx + bx^3 \qquad (2.1.2)$$

Hence (2.1.1) becomes

$$\frac{md^2x}{} + kx + bx^3 = 0 \qquad (2.1.3)$$

$$dt^2$$

or

$$\frac{d^2x}{dt^2} + \omega^2 x + \alpha x^3 = 0 \qquad (2.1.4)$$

where

$$\omega = \left(\frac{k}{m}\right)^{1/2} \qquad (2.1.5)$$

$$\alpha = \frac{b}{m} \qquad (2.1.6)$$

Equation (2.1.4) occurs in the theory of nonlinear vibrating systems and certain types of nonlinear electrical systems and serves to illustrate the general method of analysis. Let us consider the solution of Equation. (2.1.4) subject to the initial conditions.

$$\left.\begin{aligned} x &= a \\ \dot{x} &= 0 \end{aligned}\right\} \text{at } t = 0 \qquad (2.1.7)$$

That is, the mass is displaced a distance *a* and allowed to oscillate freely. We are interested in studying the subsequent behavior of the motion.

Let us now multiply Equation.(2.1.4) by e^{-st} dt and integrate from 0 to ∞. We thus obtain.

$$\int_0^\infty e^{-st} \left(\frac{d^2x}{dt^2} + \omega^2 x + \alpha x^3\right) dt = 0 \qquad (2.1.8)$$

Let us use the notation

$$Lx(t) = y(s) \qquad (2.1.9)$$

Now, by integration by parts, it may easily be shown that

$$L\frac{d^2x}{dt^2} = s^2y - sx_0 - \dot{x}_0 \qquad (2.1.10)$$

where $\dot{x}_o$ and x_o are the initial velocity and displacement of the particle at t = 0.

In view of the initial conditions (2.1.7) and the Laplace transform (2.1.10) , Equation. (2.1.11) may be written in the form.

$$(s^2 + \omega^2)\, y = sa - \alpha\, Lx^3$$

Now let

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \ldots \qquad (2.1.12)$$

$$\omega^2 = \omega^2_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + \ldots \qquad (2.1.13)$$

$$y = y_o + \alpha y_1 + \alpha^2 y_2 + \alpha^3 y_3 + \ldots \qquad (2.1.14)$$

$$y_i(s) = Lx_i(t)$$

In these expressions the quanties $x_r(t)$ are functions of time to be determined, and ω_0 is the frequency , which will be determined later. The c_i quantites are constants which are chosen to eliminate resonance conditions in a manner that will become clear as we proceed. The functions $y_r(s)$are the Laplace transforms of the function $x_r(t)$.

In most nonlinear dynamical systems the quantity α is small compared with ω^2, and the series (2.1.12) may be shown to converge. In the following discussion let us limit our calculations by omitting all the terms containing α to a power higher than the third. Substituting the above expressions into (2.1.11), we obtain

$$s^2(y_0 + \alpha y_1 + \alpha^2 y_2 + \alpha^3 y_3)$$
$$+(\omega^2{}_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3)(y_0 + \alpha y_1 + \alpha^2 y_2 + \alpha^3 y_3)$$
$$= sa - \alpha L(x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3)^3 \qquad (2.1.16)$$

If we now neglect all terms containing α to powers higher than the third, we obtain.

$$(s^2 y_o + \omega_o{}^2 y_o) + \alpha(s^2 y_1 + \omega_o{}^2 y_1 + c^1 y_o + Lx_o{}^3)$$
$$+ \alpha^2(s^2 y_2 + \omega^2{}_o y_2 + c_2 y_o + c_1 y_1 + L3x_o{}^2\, x_1)$$
$$+\alpha^3\left[s^2 y_3 + \omega^2{}_o y_3 + c_3 y_o + c_2 y_1 + c_1 y_2 + L(3x^2{}_o x^2 + 3x_o x^2{}_1)\right] = sa$$
$$(2.1.17)$$

This equation must hold for any value of the quantity α. This means that each factor for each of the three powers of α must be zero. Hence Equation (2.1.17) splits up into the following system of equations:

$$s^2 y_o + \omega_o{}^2 y_o = sa \qquad (2.1.18)$$

$$s^2 y_1 + \omega_o{}^2 y_1 = -c_1 y_o - Lx^3{}_o \qquad (2.1.19)$$

$$s^2 y_2 + \omega_o{}^2 y_2 = -c_2 y_o - c_1 y_1 - L3x^3{}_o x_1 \qquad (2.1.20)$$

$$s^2 y_3 + \omega_o{}^2 y_3 = -c_3 y_o - c_2 y_1 - c_1 y_2 - L(3x^2{}_o x + 3x_o x^2{}_1) \qquad (2.1.21)$$

Using the notation

$$T(\phi) = \frac{1}{s^2 + \phi^2} \qquad (2.1.22)$$

Eq.(2.1.18) may be written in the form

$y_o = saT(\omega_o) = Lx_o$

From the table of transforms (see p.617) [3] ,we have

$$L^{-1}\, saT(\omega_o) = a\cos \omega_o t = x_o \qquad (2.1.24)$$

This represents the first approximation to the solution of Eq.(2.1.4) Subject to these initial conditions (2.1.7). The transform of the second approximation as given by (2.1.19) may be written in the form.

$$y_1 = -c_1 y_o T(\omega_o) - T(\omega_o)\, Lx^3_o \qquad (2.1.25)$$

From the table of transforms [3] we have

$$Lx^3_o = La^3 \cos^3\omega_o t = \frac{a3}{4}\left(sT(\omega_o)\, T(3\omega_o)\right)$$

Substituting (1.1.23) and (1.1.26) into (1.1.25), we obtain

$$y_1 = -sT^2(\omega_o)\left(c_1 a + \frac{3a3}{4}\right) - \frac{a3}{4}\, sT(\omega_o)\, T(3\omega_o)$$

Now, from the table of transform [3] , it is seen that

$$L^{-1} sT^2(\omega_o) = \frac{t}{2\omega_o} \sin \omega_o t \qquad (2.1.28)$$

Hence the first term of the right-hand member of(2.1.27) corresponds to a condition of resonance. We may eliminate this condition of resonance by placing the coefficient of this term equal to zero. Then

$$c_1 a + \frac{3a^3}{4} = 0 \qquad (2.1.29)$$

Or

$$c_1 = -\frac{3a^2}{4} \qquad (2.1.30)$$

This determines the constant c_1. With the resonance condition eliminated, (2.1.27) reduces to

$$y_1 = -\frac{a3}{4} sT(\omega_o)T(3\omega_o) \qquad 2.1.31$$

Making use of the table of transforms [3] we obtain second approximation.

$$x_1 = L^{-1}y_1 = L^{-1}\left(\frac{-a^3}{4} sT(\omega_o)\, T(3\omega_o)\right)$$

$$= \frac{a^3}{32\omega^2_o}(\cos 3\omega_o t - \cos \omega_o t) \qquad 2.1.32$$

If we limit our calculations to the second approximation, we obtain from (2.1.12) , (2.1.24), and (2.1.32).

$$x = a\cos \omega_o t + \frac{\alpha a^3}{32\omega_o^2}(\cos 3\omega_o t - \cos \omega_o t) \qquad 2.1.33$$

The angular frequency is obtained by substituting the value of c_1 given by (2.1.30) into (2.1.13). This gives

$$\omega^2_o = \omega^2 + ¾\, a^2\alpha \qquad 2.1.34$$

From this we see that the presence of the term x^3 in the equation introduces a higher harmonic term $\cos 3\omega_o t$ and the fundamental frequency is not constant but depends on the amplitude *a* and increases with a provided that the quantity α is positive.

The third approximation is obtained by substituting the above values of y_o, y_1, x_o, and x_1 into (2.1.21). This gives.

$$y_2 = -c_2 s a T^2(\omega_o) + \frac{c_1 a^3}{4} T(\omega_o)\, sT(\omega_o)\, T(3\omega_o) - T(\omega_o)L(3x_o^2 x_1)$$

(2.1.35)

We must now compute

$$L(3x^2{}_o x_1) = L\left(\frac{3a^5}{32\omega^2{}_o}\cos^2\omega_o t\cos 3\omega_o t - \cos^3\omega_o t\right) \quad 2.1.36)$$

By using the table of transforms we easily obtain

$$L(3x^2{}_o x_1) = L\left(\frac{3a^5 s}{(4\times 32)\omega^2{}_o} T(5\omega_o) + T(3\omega_o) + T(3\omega_o) - 2T(\omega_o)\right)$$

(2.1.37)

Substituting this value of $L(3x_o{}^2x_1)$ and making use of the identity

$$T(a)\,T(b) = \frac{1}{b^2 - a^2}(T(a) - T(b)) \quad (2.1.38$$

We write (1.1.35) in the form

$$y_2 = sT^2(\omega_o)\left(- c_2 a + \frac{c_1 a^3}{32\,\omega^2{}_o} + \frac{3a^5}{64\,\omega^2{}_o}\right)$$

$$+ sT(\omega_o)\,T(3\omega_o)\left(-\frac{c_1 a^3}{32\,\omega^2{}_o} - \frac{3a^5}{(4\times 32)\,\omega^2{}_o}\right)$$

$$- sT(\omega_o)T(5\omega_o)\frac{3a^5}{(4\times 32)\omega^2{}_o} \quad (2.1.39)$$

To eliminate the condition of resonance, we equate the coefficient of the $sT^2(\omega_o)$ term to zero, substituting the value of c_1 into the coefficient. We thus obtain.

$$c_2 = \frac{3a^4}{128\omega^2{}_o} \quad (2.1.40)$$

On substituting the value of c_1 into the second member of (2.1.39), we see that this term vanishes and we have.

$$y_2 = -\frac{3a^5}{128\omega^2_o} sT(\omega_o)\, T(5\omega_o) \qquad (2.1.41)$$

Using the table of transforms to obtain the inverse transform of y_2, we have the third approximation.

$$x_2 = \frac{a^5}{1.024\omega^4_o}(\cos 5\omega_o t - \cos \omega_o t) \qquad (2.1.42)$$

From (1.1.12) we thus have the third approximation

$$x = a \cos \omega_o t + \frac{\alpha a^3}{32\omega^2_o} (\cos 3\omega_o t - \cos\omega_o t)$$

$$+\frac{\alpha^2 a^5}{1.024\omega^4_o}(\cos 5\omega_o t - \cos \omega_o t) \qquad (2.1.43)$$

where now the fundamental frequency is given by (2.1.13) as

$$\omega^2_o = \omega^2 + \tfrac{3}{4} a^2 \alpha - \frac{3a^4 \alpha^2}{128\omega^2_o} \qquad (2.1.44$$

The fourth approximation is obtained in the same manner from (2.1.21).

We compute

$$L(3x^2_o x_2 + 3x_o x^2_1) = \frac{3a^7 s}{(4 \times 1.024)\, \omega^4_o}(2T(7\omega_o) + T(5\omega_o) - 3T(3\omega_o)) \qquad (2.1.45)$$

By using the table of transforms. Substituting the values of the quantities y_o , y_1 and y_2 as given above into Equation. (2.1.21) and making use of the relation (2.1.45), we obtain.

$$y_3 = -\, c_3\, s a T^2(\omega_o) - \frac{c_2 T(\omega_o)\, a^3 s}{32\omega 2_o}\left[T(3\omega_o) - T(\omega_o)\right]$$

$$\frac{-\,c_1T(\omega_o)a^5s}{1{,}024\omega^4_o}\left[(T(5\omega_o)-T(\omega_o)\right]$$

$$-\frac{3a^7\,sT(\omega_o)}{(4\times 1{,}024)\omega^4_0}\left[2T(7\omega_o)+T(5\omega_o)-3T(3\omega_o)\right] \qquad (2.1.46)$$

The condition for no resonance leads to

$$c_3a=\frac{c_2a^3}{32\omega^2_0}+\frac{c_1a^5}{1.024\omega^4_0} \qquad (2.1.47)$$

Subsisting the values of c_1 and c_2 given by (2.1.30) and (2.1.40), into (2.1.47), we obtain

$$c_3 = 0 \qquad (2.1.48)$$

Suppressing the resonance terms, Equation(2.1.46) reduces to

$$y_3 = sT(\omega_o)T(3\omega_o)\frac{3a^7}{2.048\omega^4_0}+\ sT(\omega_o)T(7\omega_o)\left(-\ \frac{6a^7}{4.096\omega^4_0}\right)$$

(2.1.50)

Computing the inverse Laplace transform by the use of the table of transforms [3] , we obtain.

$$x_3 = \frac{a^7}{32.768\omega^6_0}\,(5\cos\omega_o t-3\cos 3\omega_o t+\cos {}^7\omega_o t)$$

(2.1.51)

Substituting x_o, x_1, x_2, x_3 into (2.1.12), we obtain the fourth approximation

$$x = a\cos\omega_o t+\frac{\alpha a^3}{32\omega^2_0}(\cos 3\,\omega_o t-\cos\omega_o t)$$

$$+\frac{\alpha^2 a^5}{1.024\omega^4_0}(\cos 5\omega_o t - \cos \omega_o t)$$

$$+\frac{\alpha^3 a^7}{32.768\omega_o^6}(5 \cos\omega_o t - 3\ \omega_o t + \cos 7\ \omega_o t) \qquad (2.1.52)$$

To this approximation the fundamental frequency ω_o is given by and is

$$\omega^2_o = \omega^2 + \tfrac{3}{4}\alpha a^2 - \frac{3}{128}\alpha^2 \frac{a^4}{\omega^2_o} \qquad (2.1.53)$$

Now, in all the calculations, terms to higher power than the third have been omitted. We may simplify Equation. (2.1.53) by substituting on the right-hand side the value of ω^2_o given by(2.1.34). We thus obtain.

$$\omega^2_o = \omega^2 + \tfrac{3}{4}\ \alpha a^2 - \frac{3}{128}\alpha 2 \left(\frac{a^4}{\omega^2 + \tfrac{3}{4}\alpha a^2}\right) \qquad (2.1.54)$$

Expanding the term in parentheses in powers of α and retaining powers of α only up to the third, we have

$$\omega^2_o = \omega^2 + \tfrac{3}{4}\ \alpha a^2 - \frac{3\alpha^2}{128}\ \frac{a^4}{\omega^2} + \frac{9\alpha^3 a^6}{512\omega^4}$$

Further approximation may be carried out by the same general procedure.

The vibrations of a pendulum:

The above analysis may be applied to the study of a theoretical pendulum the equation of motion of such a pendulum is

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \qquad (2.1.55)$$

Where l is the length of the pendulum and g is the gravitational constant. If we develop sin θ in a power series in θ and retain only the first two terms of the series, we obtain.

$$\ddot{\theta} + \frac{g}{l}\theta - \frac{g}{6l}\theta^3 = 0 \qquad (2.1.56)$$

If we stipulate the initial condition that, that, at t = 0, $\theta = \theta_o$, we may make use of the preceding analysis by letting.

$$\theta_o = a \qquad (2.1.57)$$

$$\omega^2 = \frac{g}{l} \qquad 2.1.58$$

$$\alpha = \frac{g}{6l} \qquad (2.1.59)$$

Using the first approximation for the angular frequency as given by (2.1.34), we have

$$\omega^2_0 = \frac{g}{I} - \frac{g\,\theta^2_0}{8I} \qquad (2.1.60)$$

To this approximation the period of the oscillation is given by.

$$T = \frac{2\pi}{\omega_o} \qquad \frac{2\pi}{\left[(g/I)(\theta^2_0/8)\right]^{1/2}} \qquad (2.1.61)$$

For small amplitude θ_0 the radical nay be expanded in powers of θ_0, and we may write

$$T = 2\pi\left(\frac{I}{g}\right)^{1/2}\left(1 + \frac{\theta^2_0}{16}\right) \qquad (2.1.62)$$

This formula gives excellent results for small amplitudes of oscillation.

2.2 Forced vibration of nonlinear systems:

In the last section free vibrations of oscillating system with nonlinear restoring forces were considered. In this section the forced oscillations of system with nonlinear restoring forces will be considered.

As a typical example of such a system, let us consider the mechanical case of an oscillator of mass m acted upon by a nonlinear elastic restoring force F(x) and by a periodic external force $F_o \cos \omega t$. The equation of motion of such a system is

$$m\frac{d^2x}{dt^2} + F(x) = F_o \cos \omega t \qquad (2.2.1)$$

Let us first discuss the case in which the restoring force is symmetric, that is, has equal magnitude at corresponding points on both sides of the position of equilibrium or position of rest. In this case only odd powers may occur in the law of force. Otherwise we have an unsymmetrical law of force and hence an unsymmetric vibration. This is expressed mathematically by the condition.

$$F(-x) = F(x) \qquad (2.2.2)$$

Since the methods of analysis and the qualitative results do not depend greatly upon the special form of F(x), we shall choose the following form for the restoring force F(x),.

$$F(x) = kx - \delta x^3 \qquad (2.2.3)$$

Where $k > 0$.

If $\delta > 0$, it is said that the restoring force corresponds to a soft spring while if $\delta < 0$, the restoring force is said to correspond to a hard spring. Is the case that $\delta > 0$ the restoring force decreases with the amplitude of oscillatios as in the case of a pendulum. In this case the natural frequency decreases with increasing amplitude.

Inserting (2.2.3) into (2.2.1), we have the equation of motion

$$m\frac{d^2x}{dt^2} + kx - \delta x^3 = F_o \cos \omega t \qquad (2.2.4)$$

This equation is known in the literature as Duffuing's equation.

Experiments performed on dynamical systems whose equation of motion are of the form(2.2.4)show that as the time t increases, the motion of the system becomes periodic after some transient motions have died out. The period of the resulting oscillations is found to have a fundamental frequency of $\omega/2\pi$ and may therefore be represented by a Fourier series in multiples of ω.

The amplitude of the steady state (as $t \to \infty$) may be calculated by the following approximate method. As a first approximation let us assume.

$$x_1 = a \cos \omega t \qquad (2.2.5)$$

where the amplitude a is to be determined. If we substitute this expression for x in (2.2.4) and make use of the trigonometric identity.

$$\cos^3 \omega t = \tfrac{1}{4} \cos 3\omega t + \tfrac{3}{4} \cos \omega t \qquad (2.2.6)$$

We obtain the equation

$$m a\omega^2 + k a - \tfrac{3}{4}\delta a^3 - \frac{\delta\, a3}{4} \cos 3\omega t = F_o \cos \omega t \qquad (2.2.7)$$

If the fundamental vibration is to satisfy Eq. (2.2.4), we must have

$$\frac{3\delta}{4m} a^3 + (\omega^2 - \omega^2_o)\, a + \frac{F_o}{m} = 0 \qquad (2.2.8)$$

Where

$$\omega_o = \sqrt{\frac{k}{m}} \qquad (2.2.9)$$

This is the natural angular frequency of the system in the absence of the nonlinear term.

Equation (2.2.8) determines the amplitude of the oscillation. If we divide Equation (2.2.8) by ω^2_0, we obtain

$$\frac{3\delta a^3}{4m\, \omega^2_0} = \left(1 - \frac{\omega^2}{\omega^2_0}\right) a - \frac{F_0}{\omega^2_0 m} \qquad (2.2.10)$$

The roots of this cubic equation in a may be obtained graphically by constructing y-a coordinate system, as shown in Figure (1). This figure represents the cubical parabola.

$$y = \frac{3\delta a^3}{4m\omega_o^2} \qquad (2.2.11)$$

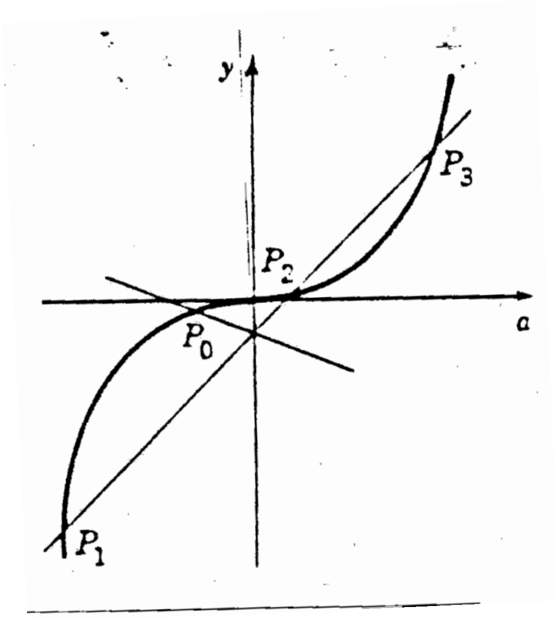


Figure. (1)

and the straight line

$$y = \left(1 - \frac{\omega^2}{\omega_0^2}\right) a - \frac{F_0}{\omega_0^2 m} \qquad (2.2.12)$$

The possible values of a are the abscissa of the points of intersection of these curves.

If ω is large, the slope of the straight line is negative and there is only one point of intersection P_o. There is also only one point of intersection for $\omega = \omega_o$. If now ω decreases, the straight line rotates until it intersects the cubical parabola at three points P_1, P_2, and P_3. The abscissa of these points corresponds to three possible amplitudes. The amplitude-versus-frequency curve has the form shown in Figure.(2)

A more precise analysis shows that if we approach from the low frequency side, the amplitude corresponding to the lower branch is the stable -one. As ω increases, we arrive at the limiting point G. As a continues to increase beyond this point, only the upper branch yields a real point of intersection in Figure.(1). It is then seen that with a continuous increase of ω, the amplitude a will suddenly jump from the lower branch to the upper one at G.

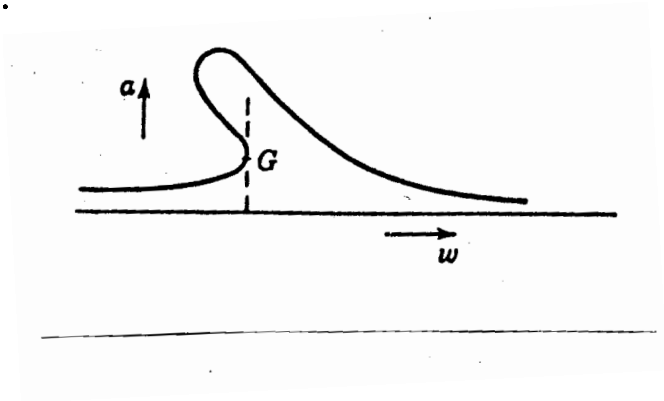


Figure (2)

These discontinuities or jumps in amplitude are frequently observed in nonlinear vibration processes both electrical and mechanical.

The higher approximations:

It will now be shown how the next approximations of the motion are obtained. If Equation.(2.2.4) is solved for d^2x/dt^2 and if the first approximation(2.2.5) is substituted in the right-hand member for x, we obtain.

$$\frac{d^2x}{dt^2} = \frac{F_0}{m}\cos\omega t - \frac{k\,a}{m}\cos\omega t + \frac{3\delta\,a^3}{4m}\cos\omega t + \frac{\delta\,a^3}{4m}\cos 3\omega t \qquad (2.2.13)$$

Making use of (2.2.8), this reduces to

$$\frac{d^2x}{dt^2} = -\omega^2 a\cos\omega t + \frac{\delta\,a^3}{4m}\cos 3\omega t \qquad (2.2.14)$$

Integration gives

$$x_2 = a \cos \omega t - \frac{\delta a^3}{36m\omega^2} \cos 3\omega t = a \cos \omega t \; \frac{\delta a^3 \omega^2_o}{36k\omega^2} \cos 3\omega t \qquad (2.2.15)$$

This second approximation may then be substituted into (2.2.4) to obtain the third approximation. In this manner any number of terms of the Fourier-series solution may be obtained. The investigation of the convergence of the process shows that the series obtained converges if δ is small.

The case of an unsymmetrical restoring force:

If we add a quadratic term to the elastic restoring force so that

$$F(x) = kx + \delta x^2 \qquad (2.2.16)$$

Then the vibration becomes unsymmetric since changing the sign of x does not change that of the quadratic term, and hence the restoring force has different values at two points that are symmetric with respect to the origin. The equation of motion is now.

$$m\frac{d^2x}{dt^2} + kx + \delta x^2 = F_o \cos \omega t \qquad (2.2.17)$$

In this case we assume

$$x_1 = a \cos \omega t + b \qquad (2.2.18)$$

As the first approximation. The constant b is introduced to allow for the lack of symmetry. We insert this approximation into(2.2.17) and determine a and b in such a way that the constant term and the fundamental vibration satisfy the differential equation.

Using the trigonometric identity

$$\cos^2 \omega t = \frac{1 + \cos 2\omega t}{2} \qquad (2.2.19)$$

We obtain the two equations

$$b^2 + \frac{k}{\delta} b + \frac{a^2}{2} = 0 \qquad (2.2.20)$$

and

$$a(\omega^2 - \omega^2_o) \frac{2\delta a b}{m} + \frac{F_0}{m} = 0 \qquad (2.2.21)$$

If δ is small, we have from (2.2.20)

$$b = \frac{\delta a^2}{2k} \qquad (2.2.22)$$

If we now substitute this into (2.2.21), we have

$$\frac{\delta^2}{km} a^3 + a(\omega^2 - \omega^2 o) + \frac{F_o}{m} = 0 \qquad (2.2.23)$$

This is a cubic equation for the amplitude a. It may be solved graphically, and it is found that under certain conditions it has three roots so that the "jump" phenomenon occurs here as in the case of the symmetrical vibrations. The higher approximations are obtained in the same manner as in the symmetrical case.

Sub harmonic response:

Periodic solutions of the Doffing equation (2.2.4) have been considered. These solutions have a fundamental period $P = 2\pi/\omega$ equal to the period of the external exciting force. Experiments show that permanent oscillations with a frequency of ½,1/3 , …, 1/n of that of the applied force can occur in nonlinear systems. This phenomenon is called subharmanic resonance.

It is known that in linear systems having damping the permanent oscillations of the system have a frequency exactly equal to that of the exciting force, and hence subharmonic resonance is impossible in linear systems. In nonlinear systems, however, even with damping present, the phenomenon of subharmonic resonance is exhibited.

The usual explanation offered of the phenomenon of subharmonic resonance is that the oscillations of a nonlinear system contain higher harmonics in profusion. It is therefore possible that an external force with a frequency the same as one of the higher harmonics may be able to sustain and excite harmonics of lower frequency. This of course requires cretin conditions to be true of the system. The mathematical discussion of the problem of subharmonic resonance is a matter of some difficulty. As an example of a typical investigation of the possibility of subharmonic response, consider the following nonlinear differential equation:

$$\frac{d^2x}{dt^2} + w^2{}_0 x + bx^3 = \frac{F_0}{m}\cos wt \qquad (2.2.24)$$

Assume that a possible solution of this equation has the following form.

$$x = A_o \cos\frac{wt}{3}$$

If this assumed form of the solution is substituted into (2.2.24), the result may be written in the following form:

$$\left(w^2{}_o A_o - \frac{w^2}{9} A_o + \tfrac{3}{4} b A_o{}^3\right)\cos\frac{wt}{3} + \frac{b}{4} A_o{}^3 \cos wt = \frac{F_o}{m}\cos wt \qquad (2.2.25$$

For (2.2.25) to satisfy (2.2.24) we must therefore have

$$w^2{}_o A_o - w^2\frac{A_o}{9} + 3b\frac{A_o{}^3}{4} = 0 \qquad (2.2.26)$$

And

$$\frac{b}{4} A^3{}_o = \frac{F_o}{m} \qquad (2.2.27)$$

Hence an oscillation of the type (2.2.25) with an amplitude of

$$A_o = \left(\frac{4F_o}{mb}\right)^{1/2}$$

is possible provided the angular frequency of the forcing function satisfies the equation.

$$w^2 = 9w^2_o + b\frac{27}{4}\left(\frac{4F_o}{mb}\right) \qquad (2.2.28)$$

The stability of this solution requires a separate investigation [3].

2.3 Forced Oscillations with Damping:

In many practical problems, particularly in the field of electrical circuit theory, approximate solutions of nonlinear differential equations must be determined in which the forcing function contains a constant plus a harmonic term and the circuit contains damping. The following equation is typical of this class of differential equations:

$$L\frac{di}{di} + Ri + S_o q + aq^3 = E_o + E_m \sin(wt + \theta) \qquad (2.3.1)$$

This equation governs the oscillations of an electrical circuit consisting of an inductance L and a resistance R, both of which are linear, in series with a nonlinear capacitor. S_o is the constant initial elastance of this capacitor, and *a* is a constant which is the measure of the departure of the capacitor from linearity. In practical circuits *a* is a small positive number. The functions i and q are the current of the circuit and the charge separation on the plates of the capacitor, respectively. The functions are related by the differential equation

$$i = \frac{dq}{dt} \tag{2.3.2}$$

In order to obtain an approximate steady- state solution by Duffing's method, a periodic solution of the following form is assumed for (2.3.1),

$$j = I_m \sin wt \tag{2.3.3}$$

$$q = Q_o - \frac{1_m}{w} \cos wt \tag{2.3.4}$$

Where Q_o is the constant charge accumulation on the plates of the capacitor and I_m is the maximum amplitude of the alternating current of the circuit. Substitute these expressions into (2.3.1), and let:

$$F(t) = \left[L\frac{di}{dt} + Ri + S_0 q + aq^3\right] \; i = 1_m \sin wt \quad q = Q_0 - \frac{I_m}{w}\cos wt$$

$$= \left[S_0 Q_0 + aQ^3_0 + \frac{3aQ^3_0 I^2_m}{2w^2}\right] + R1_m \sin wt$$

$$+ \left[wL1_m - S_0\frac{I_m}{w} - \frac{3a1^3_m}{4w^3} - 3aQ^2_0\frac{1_m}{w}\right]\cos wt$$

$$+ \frac{3Q_o a1^2_m}{2w^2}\cos 2wt - \frac{a1^3_m}{4w^3}\cos 3wt \tag{2.3.5}$$

With this notation Equation. (2.3.1) becomes

$$F(t) = E_0 + E_m \cos\theta \sin wt + E_m \sin\theta \cos wt \tag{2.3.6}$$

The undetermined coefficients I_m and Q_o and the phase angle θ may be adjusted to make (2.3.3) and (2.3.4) an approximate solution of (2.3.1) by equating the constant term and the coefficient of sin wt

and cos wt to the right- hand member of (2.3.6). This procedure leads to the following three equations:

$$S_0Q_0 + aQ_0\left(Q^2_0 + \frac{31^2_m}{2w^2}\right) = E_0 \qquad (2.3.7)$$

$$R1_m = E_m\cos\theta \qquad (2.3.8)$$

$$1_m\left(wL - \frac{S_0}{w}\right) - 3a\frac{1_m}{w}\left(Q^2_0 + \frac{31^2_m}{4w^2}\right) = E_m \sin\theta \qquad (2.3.9)$$

The above equations are three simultaneous equations for the determination of the unknowns I_m , Q_o, and the phase angle θ. If numerical values for the parameters involved are available, a solution of these equations may be affected by a graphical procedure similar to that outlined in section before.

In practical applications, however, considerable simplification may be introduced because the parameter a is a very small number. Since a is small, the second term of (2.3.7) may be neglected and the following approximate value for Q_o obtained,

$$Q_0 + \frac{E_0}{S_0} = C_0E_0 \qquad (2.3.10)$$

where $C_0 = 1/S_0$ is the initial capacitance of the nonlinear capacitor. If the direct potential E_0 is large in comparison with the maximum value of the harmonic potential so that $E_0 \gg E_0$, then, for the frequencies ordinarily used in practice, we have

$$Q^2_0 >> \frac{1^2_m}{2w^2} \qquad (2.3.11)$$

If the term $I^2_m/4$ is neglected in (2.3.9), this equation may be written in the form:

$$1_m \left[wL - \frac{1}{w}(S_0 + 3aC^2{}_0E^2{}_0) = E_0 \sin\theta \right. \qquad (2.3.12)$$

Hence, if (2.3.8) and (2.3.12) are squared and the results added, we obtain

$$1^2{}_m \left\{R^2 + \left[wL - \frac{1}{w}(S_0 + 3aC^2{}_0E^2{}_0)\right]^2\right\} = E^2{}_m \qquad (2.3.13)$$

and the amplitude of the alternating current of the system is given by

$$I_m = \frac{E_m}{\sqrt{R^2 + [wL - \frac{1}{w}(S_0 + 3aC^2{}_0E^2{}_0)]^2}} \qquad (2.3.14)$$

It is thus apparent that the effect of the direct potential is to increase the effective elastance of the system by an amount $3aC^2{}_0E^2{}_0$. The tangent of the phase angle θ may be determined by the division of (2.3.12) by (3-8). This procedure yields

$$\tan\theta = \frac{wL - \left(\frac{1}{w}\right)(S_0 + 3aC^2{}_0E^2{}_0)}{R} \qquad (2.3.15)$$

The principal harmonics of the current i may be obtained by writing (2.3.8) in the form

$$R\boldsymbol{i} = E_0 + E_m \sin(wt + \theta) - (L \frac{di}{dt} + S_0q + aq^3) \qquad (2.3.16)$$

And substituting (2.3.3) and (2.3.4) into the right- hand member of (2.3.16). If this is done and Equations (2.3.7), (2.3.8) , and

(2.39) used to simplify the resulting expression, the result yields the following value for the current i :

$$i = 1_m \sin wt + \frac{a 1^3{}_m}{4w^3 R} \cos 3wt - \frac{3a Q_0 1^2{}_m}{2w^2 R} \cos 2wt -$$

Higher- order harmonics may be obtained by substituting (2.3.17) into the right- hand member of (3-16) and repeating the procedure. It may be noted that, if a = 0, the system becomes a linear one and denominator of (2.3.14) reduces to the ordinary impedance of a linear series circuit.

2.4 **Forced oscillations of A nonlinear inductor:**

An important technical problem is the computation of the amplitude and phase of the fundamental and the harmonic content of the steady – state current in a circuit of the type depicted by Figure.(3).

This circuit contains a harmonic potential $E_m \sin(\omega t + \theta)$ and a bias potential E_o in series with a linear resistor and a nonlinear inductor. The nonlinear inductor consists of a coil of N turns wound on a magnetic core having a cross – sectional area A and a mean length s.

The differential equation that determines the circuit current i is

$$Ri + N\phi = E_o + E_m \sin(\omega t + \theta) \qquad (2.4.1)$$

Where ϕ is the magnetic flux in the core of the nonlinear inductor. The flux ϕ is related to the current i by Amper's law which, expressed in suitable units, is

$$H = \frac{Ni}{s} \qquad (2.4.2)$$

Where H = magnetic intensity of the core

s = mean length of the magnetic path in the core

For many practical purposes, the magnetization curve of the core material of the inductor may be represented by the following third – degree polynomial:

$$B = \mu_o H - kH^3 \qquad (2.4.3)$$

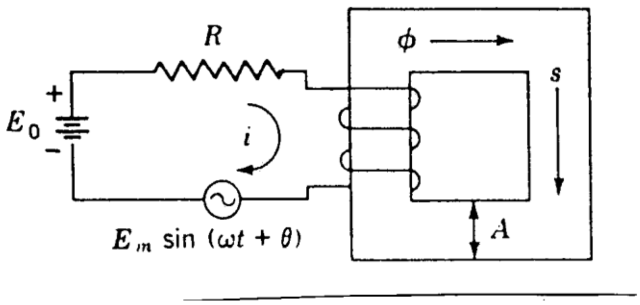


Figure. (3)

In this expression B is the magnetic induction and k is a constant determined empirically by adjusting Equation. (2.4.3) to fit the actual magnetization curve of the material of the core. μ_o is the initial permeability of the core material. It is defined by the equation

$$\mu_o = \left(\frac{dB}{dH}\right)_{H=0} \qquad (2.4.4)$$

If A is the mean cross – sectional area of the inductor core, the flux ϕ may be expressed in the form

$$\phi = AB = A(\mu_o H - kH^3) \; \frac{N}{s} i\mu_o - kA\left(\frac{Ni}{s}\right)^3 \qquad (2.4.5)$$

$$N\phi = \frac{AN^2 \mu_0}{s} i - \frac{kAN^4 i}{s^3} i^3 \qquad (2.4.6)$$

It is convenient to write this expression in the following form:

$$N\phi = L_0 i - bi^3 \qquad (2.4.7)$$

Where

$$L_o = \frac{\mu_0 N^2 A}{s} \qquad b = \frac{kAN^4}{s^3} \qquad (2.4.8)$$

L_o is the initial inductance of the nonlinear inductor. If $N\phi$ as given by Equation.(2.4.7) is substituted into (2.4.1), the result is

$$L_o \frac{di}{dt} + Ri - b\frac{d}{dt} i^3 = E_o + E_m \sin(\omega t + \theta) \qquad (2.4.9)$$

To determine the steady-state response of the nonlinear inductor, it is necessary to determine a periodic solution of Equation.

(2.4.9). The method of undetermined coefficients suggests that a periodic solution of the following form be assumed for the current:

$$i_o = I_o + I_m \sin \omega t \qquad (2.4.10)$$

Where I_o= undetermined dc component of the steady- state current

I_m= the undetermined amplitude of the fundamental of the steady – state alternating current of the circuit.

To determine I_o and Im, substitute Equation.(2.4.10) into the left member of (2.4.9) and write.

$$F(t) = L_o \frac{di_o}{dt} + Ri_o - b\frac{d}{dt} i^3_o$$

$$= RI_o + RI_m \sin \omega t + (\omega L_o I_m - \tfrac{3}{4} b\omega I^3{}_m \omega) \cos\omega t$$

$$- 3bI^2{}_o I_m \omega - (3\omega I_o I^2{}_m b) \sin 2\omega t + \tfrac{3}{4} bI^3{}_m \omega \cos 3\omega t \quad (2.4.11)$$

F(t) represents the potential drop that would exist across the circuit elements if the current flowing through the circuit had the form of Equation.(2.4.10). Since the impressed potential of the circuit has the form.

$$E(t) = E_o + E_m \sin(\omega t + \theta) = E_o + E_4 \sin \omega t + E_2 \cos \omega t \quad (2.1.12)$$

Where

$$E_1 = E_m \cos\theta \quad E_2 = E_m \sin\theta \quad (2.4.13)$$

It is evident that $F(t) \neq E(t)$ and that Equation.(2.4.10) cannot be adjusted to give the exact solution of the differential equation (2.4.9). However, an approximate solution of practical utility may be obtained by requiring that the constant term, the sine term, and the cosine term of F(t) be made equal to E(t). This stipulation leads to the following three equations:

$$RI_o = E_o \quad (2.4.14)$$

$$RI_m = E_1 = E_m \cos\theta \quad (2.4.15)$$

$$\omega L_o I_m - 3b\omega I_m \frac{I^2_m}{4 + I^2_o} = E_2 = E_m \sin\theta \qquad (2.4.16)$$

These three simultaneous equations serve to determine the unknown amplitudes I_o and I_m and the phase angle θ between the applied harmonic potential and the fundamental of the resulting alternating current of the circuit. Equation (2.2.14) gives the following value for the direct component of the current:

$$I_o = \frac{E_o}{R} \qquad (2.4.17)$$

The amplitude of the alternating current I_m may be obtained by squaring Equations. (2.4.15) and (2.4.16) and adding the result. This procedure gives

$$I^2_m \left[R^2 + \omega^2 (L_o - \tfrac{3}{4} b I^2_m - 3bI^2_o)\right] = E^2_m \qquad (2.4.18)$$

This is a cubic equation in I^2_m and can be solved by a graphical construction for a given frequency ω. In general Equation. (2.4.18) will have either one or three real roots for the amplitude I_m. The possibility of different amplitudes may lead to "jump phenomena" in special cases.

If the bias potential is large so that the direct current I_o is also large, it may be assumed that.

$$\frac{I^2_m}{4} < I^2_o \qquad (2.4.19)$$

If the term $I^2_m/4$ is neglected in Equation. (2.4.18), this equation can be solved for the amplitude I_m directly. The result is as follows.

$$Im = \frac{E_m}{\sqrt{R^2 + \omega^2(L_o - 3bI^2_o)^2}} \qquad (2.4.20)$$

Or

$$I_m = \frac{E_m}{\sqrt{R^2 + \omega^2(L_o - 3(b/R^2)\,E^2_o)^2}} \qquad (2.4.21)$$

Equation (2.4.21) shows that the effect of increasing the bias potential is to decrease the effective inductance of the circuit and hence to increase the amplitude of the alternation current. This indicates that (by) changing the magnitude of the biasing potential is possible to effect a considerable change in the amplitude I_m of the alternating current of the circuit.

The tangent of the phase angle θ of the alternating current of the nonlinear inductor may be obtained by means of Equations.(2.4.15) and (2.4.16)in the form.

$$\tan\theta = \frac{E_2}{E_1} = \frac{\omega}{R}\; L_o - \frac{3bE^2_o}{R^2} \qquad (2.4.22)$$

To this degree of approximation, it is seen that the nonlinear inductor circuit behaves as if it were a linear circuit that has a resistance R and an inductive reactance $X_L = \omega(L_o - 3bE^2_o/R^2)$.

Chapter three

The dynamic systems and their periodicity orbits

3.1.The Linear System $\dot{x} = Ax$

We first review some features of the linear system

$$\frac{dx}{dt} \stackrel{def}{=} \dot{x} = Ax, \quad x \in R^n \qquad (3.1.1)$$

where A is an n x n matrix with constant coefficients. For more information and background see a standard introductory text on differential equations such as Braun [1978]: for more detailed review of the linear algebra from the viewpoint of dynamical system theory, Hirsch and Smale [1974] or Arnold [1973] are recommended.

By a solution of (3.1.1) we mean a vector valued function $x(x_o, t)$ depending on time t and the initial condition

$$x(0) = x_o: \tag{3.1.2}$$

$x(x_o, t)$ is thus a solution of the initial value problem (3.1.1)-(3.1.2). In terms of the flow ϕ_t, we have $x(x_o, t) \equiv \phi_t(x_o)$. Theorem 1.0.4 [5] guarantees that the solution $x(x_o, t)$ of the linear system is defined for all $t \in R$ and $x_o \in R^n$. Note that such global existence in time dose not generally hold for nonlinear systems, as we have already seen. However, no such problems occur for (3.1.1), the solution of which is given by

$$x(x_o, t) = e^{tA} x_o, \tag{3.1.3}$$

Where e^{tA} is the n x n matrix obtained by exponentiation A. We will see how e^{tA} can be calculated most conveniently in moment, but first note that it is defined by the convergent series.

$$e^{tA} = 1 + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^n}{n!}A^n + \ldots \tag{3.1.4}$$

A general solution to (3.1.3) can be obtained by linear superposition of n linearly independent solutions $\{x^1(t), \ldots, x''(t)\}$:

$$x(t) = \sum_{j=1}^{n} c_j x^j(t), \tag{3.1.5}$$

Where the n unknown constants c_j are to be determined by initial conditions.

If A has n linearly independent eigenvectors v^j, j = 1, …,n, then we may take as a basis for the space of solutions the vector valued functions

$$x^j(t) = e^{\lambda jt} v^j,$$

Where λ_j is the eigenvalue associated with v^j. For complex eigenvalues without multiplicity, λ_j, $\lambda_j = \alpha_j \pm iB_j$, having eigenvectors $v^R \pm iv^1$, we may take.

$$x^j = e^{\alpha jt} (v^R \cos Bt - v^I \sin Bt), \qquad (3.1.6)$$

$$x^{j+1} = e^{\alpha jt} (v^R \sin Bt + v^I \cos Bt), \qquad (3.1.7)$$

as the associated pair of (real) linearly independent solutions. When there are repeated eigenvalues and less than n eigenvectors, then one generates the generalized eigenvectors as described by Braun [1978], for example. Again one obtains a set of n linearly independent solutions. We denote the fundamental solution matrix having these n solutions for its columns as

$$X(t) = [x^1(t), \ldots, x^n(t)] \; . \qquad (3.1.8)$$

The columns $x^j(t)$, j = 1, …, n of X (t) form a basis for the space of solutions of (3.1.1). It is easy to show that

$$e^{tA} = X(t)\, X(0)^{-1}., \qquad (3.1.9)$$

Example (3.1.1) Find e^{tA} for

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Solution:

$$\begin{vmatrix} 2-\lambda & 1 & 3 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(2-\lambda)(0-\lambda) - 0\right] - 1\times 0 + 3\left[0 - (2-\lambda)\right] = 0$$

$$(2-\lambda)\left[-\lambda(2-\lambda)\right] - 3(2-\lambda) =$$

$$(2-\lambda)\left[-\lambda(2-\lambda) - 3\right] = 0$$

$$(2-\lambda)\left[\lambda^2 - 2\lambda - 3\right] = 0$$

$$(2-\lambda)(\lambda^2 - 2\lambda - 3) = 0$$

$$(2-\lambda)(\lambda - 3)(\lambda + 1) = 0$$

$\therefore \lambda = 2,\ \lambda = 3,\ \lambda = 1$

As $\lambda = 2$

$$\begin{vmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = 0$$

$y + 3z = 0 \longrightarrow y = -3z$

$x - 2z = 0 \longrightarrow x = 2z$

let $z = 1 \longrightarrow y = -3,\ x = 2$

$\therefore$ The eigenvector is

$$\therefore \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 2 \\ -3 \\ 1 \end{vmatrix}$$

As $\lambda = 3$

$$\begin{vmatrix} -1 & 1 & 3 \\ 0 & -1 & 0 \\ 1 & 0 & -3 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = 0$$

$- x + y + 3 z = 0$

$-y = 0 \Rightarrow y = 0$

$x - 3z = 0$

$x = 3z$

let $z = 1 \Rightarrow x = 3$

:. The eiegen vector is

$$\therefore \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 3 \\ 0 \\ 1 \end{vmatrix}$$

As $\lambda = -1$

$$\begin{vmatrix} 3 & 1 & 3 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = 0$$

$3x + y + 3z = 0$

$3y = 0 \Rightarrow y = 0$

$x + z = 0 \Rightarrow x = -z$

let $z = 1 \Rightarrow x = -1$

:. The eiegen vector is

$$\therefore \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ 1 \end{vmatrix}$$

:. The fundamental solution is

$$X(t) = \begin{vmatrix} 2e^{2t} & 3e^{3t} & -e^{-t} \\ 3e^{2}t & 0 & 0 \\ -e^{2t} & e^{3t} & e^{-t} \end{vmatrix}$$

$$X(0^{)-1} = \begin{vmatrix} 0 & 1/4 & -1/4 \\ 1/3 & -1/12 & 5/12 \\ 0 & 1/4 & 3/4 \end{vmatrix}$$

$$\therefore e^{tA} = X(t)\,X(0)^{-1}$$

$$\therefore e^{tA} = \begin{vmatrix} e^{3t} & \frac{e^{2t}}{2} - \frac{e^{3t}}{4} - \frac{e^{-t}}{4} & \frac{-e^{2t}}{2} + \frac{5e^{3t}}{4} - \frac{-3e^{-t}}{4} \\ 0 & \frac{3e^{2t}}{4} & \frac{-3e^{2t}}{4} \\ \frac{e^{3+}}{3} & \frac{-e^{2t}}{4} - \frac{e^{3t}}{12} + \frac{e^{-t}}{4} & \frac{e^{2t}}{4} + \frac{5e^{3t}}{12} + \frac{3e^{-t}}{4} \end{vmatrix}$$

Equation (3.1.1) may also be solved by first finding an invertible transformation T which diagonalizes A or at least puts it into Jordan normal form (if there are repeated eigenvalues). Equation (3.1.1) becomes.

•

$$y = J\,y, \qquad (3.1.10)$$

Where $J = T^{-1}AT$ and $x = T\,y$, Equation (3.1.10) is easy to work with, but since the columns of T are the (generalized) eigenvectors of A, just as much work is required as in the former method. The exponential e^{tA} may be computed as

$$e^{tA} = Te^{tj}\,T^{-1} \qquad (3.1.11)$$

(cf. Hirsch and Smale [1974], pp. 84 – 87), where exponentials are evaluated for the three 2 × 2 jordan form matrices:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad e^{tA} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

$$\begin{pmatrix} \propto & -B \\ B & \propto \end{pmatrix}, \quad e^{tA} = e^{\propto t}\begin{pmatrix} \cos Bt & -\sin Bt \\ \sin Bt & \cos Bt \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}, \quad e^{tA} = e^{\lambda t}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

We also note that if v^j is an eigenvector belonging to a real eigenvalue λ_j of A, then v^j is also an eigenvector belonging to the eigenvalue e^{λ_j} of e^A. Moreover , if span $\{Re(v^j), Im(v^j)\}$ is an eigenspace belonging to a complex conjugate pair λ_j , $\bar{\lambda}_j$ of eigenvalues, then it is also an eigenspace belonging to $e^{\lambda_j}, e^{\bar{\lambda}_j}$.

3-2 Flows and Invariant Subspaces:

The matrix e^{tA} can be regarded as a mapping from R^n to R^n: given any point x_o in R^n, $x(x_o,t) = e^{tA}x_o$ is the point at which the solution based at x_o lies after time t. The operator e^{tA} hence contains global information on the set of all solutions of(3.1.1), since the formula (3.1.3) holds for all points $x_o \in R^n$. , we say that e^{tA} defines a flow on R^n and that this flow (or "phase flow") is generated by the vector field Ax defined on R^n: e^{tA} is our first specific example of a flow ϕ_t.

The flow $e^{tA}: R^n \rightarrow R^n$ can be thought of as the set of all solutions to (3.1.1). In this set certain solutions play a special role: those which lie in the linear subspaces spanned by the eigenvectors. These subspaces are invariant under $e^{tA,}$ in particular, if v^j is a (real) eigenvector of A, and hence of $e^{tA,}$ then a solution based at a point $c_j v^j \in R^n$ remains on span v^j for all time; in fact.

$$x\,(cv^j, t) = cv^j e^{\lambda jt} \qquad (3.2.1)$$

Similarly, the (two-dimensional) subspace spanned by $Re\{v^j\}$, $Im\ \{v^j\}$, when v^j is a complex eigenvector, is invariant under $e^{tA.}$ In short , the eigenspaces of A are invariant subspaces for the flow.

We divide the subspaces spanned by the eigenvectors into three classes:

The stable subspace, $E^s = span\{v^1, \dots, v^{n_s}\}$,

The unstable subspace, $E^u = span\{u^1, \dots, u^{n_u}\}$

The center subspace , $E^c = span\{w^1, \dots, w^{n_c}\}$

Where $v^1, \dots, v^{ns}$ are the n_s (generalized) eigenvectors whose eigenvalues have negative real parts, $u^1, \dots, u^{n_u}$ are the n_u

(generalized) eigenvectors whose eigenvalues have positive real parts and $w^1, \dots, w^c$ are those whose eigenvalues have zero real parts. Of course, $n_s + n_c + nu = n$. The names reflect the facts that solutions lying ^{n}on E^s are characterized by expontial decay (either monotonic or oscillatory), those lying in E^u by exponential growth, and those lying in E^c by neither. In the absence of multiple eigenvalues, these latter either oscillate at constant amplitude (if $\lambda, \lambda = \pm i\beta$) or remain constant (if $\lambda = 0$). A schematic picture appears in Figure (1) with two specific examples.

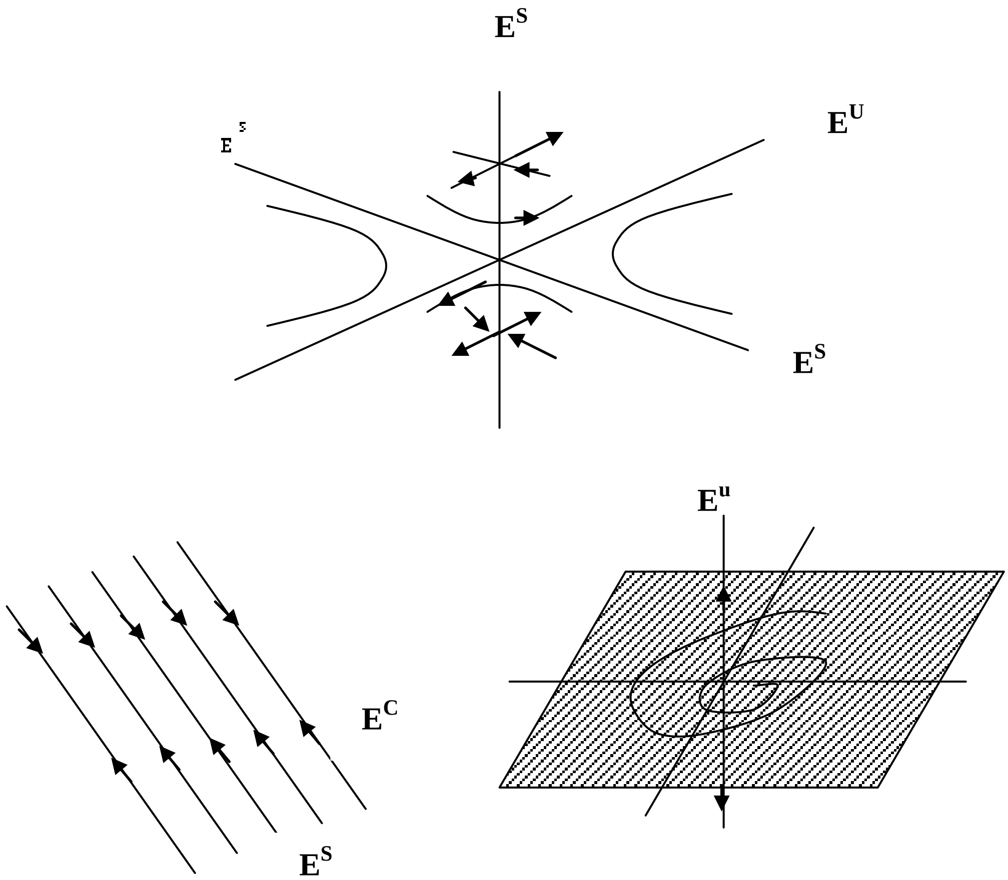


Figure (1). Invariant subspaces. (a) The three subspaces; (b)

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix},$$

(E^s = span (1, -4), E^c = span (1,0) E" = ϕ) ;   (c)

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(E^s = span (1, 0 , 0) , (1 , 1, 0) , $E^c = \phi$, E^u = (0 , 0 , 1).

When there are multiple eigenvalues for which algebraic and geometric multiplicities differ, then one may have growth of solutions in E^c.

3.3 The Nonlinear System x = *f*(x)

We must start by admitting that almost nothing beyond general statements can be made about most nonlinear systems. In the remainder of this chapter we will meet some of the delights and horrors of such systems, but we must bear in mind that the line of attack we develop only one, and that any other tool in the workshop of applied mathematics, including numerical integration, perturbation methods, and asymptotic analysis, can and should be brought to bear on a specific problem.

We recall that the basic existence – uniqueness theorem for ordinary differential equations, implies that, for smooth functions f (x), the solution to the initial value problem.

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$x = f(x)$; $x \in R^n$, $x(0) = x_o$ (3.3.1)

is defined at least in some neighborhood $t \in (-c,c)$ of $t = 0$. Thus a local flow $\phi_t: R^n \to R^n$ is defined by $\phi_t(x_o) = x(t, x_o)$ in a manner analogous to that in the linear case, although of course we cannot give a general formula like e^{tA}.

A good place to start the study of the nonlinear system $x = f(x)$ is by finding the zeros of f or the fixed point of (3.3.1). These are also referred to as zeros, equilibria, or stationary solutions. Even this may be a formidable task, although in most of our examples it will not be. Suppose then that we have a fixed point x, so that $f(x) = 0$, and we wish to characterize the behavior of solutions near $\bar{x}$. We do this by linearizing (3.3.1) at x, that is, by studying the linear system.

$$\xi = D f(\bar{x})\, \xi, \quad \xi \in R^n, \tag{3.3.2}$$

where $Df = [\partial f_i / \partial x_j]$ is the jacobian matrix of first partial derivatives of the function $f = (f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), f_n(x_1, \dots x_n))^T$ (Tdenotes transpose) and $x = \bar{x} + \xi$, $|\xi| \ll 1$.Since (3.3.2) is just a linear system of the form (3.1.1), we can do this easily . In particular, the linearized flow map $D\phi_t(x)\xi$ arising from (3.3.1) at a fixed point x is obtained from (3.3.2) by integration:

$$D\phi_t(\bar{x})\xi = e^{tDf(\bar{x})}\, \xi \tag{3.3.3}$$

The mportant question is, what can we say about the solutions of (3.3.1) based on our knowledge of (3.3.2)? The answer is provided by two fundamental result of dynamical systems theory which we give below, and may be summed up by saying that local behavior (for$|\xi|$small)does carry over in certain"nice" cases 5 .

Theorem (3.3.1) (Hartman – Grobman). If $Df(\bar{x})$ has zero or purely imaginary eigenvalues then there is a homeomorphism h defined on some neighborhood U of x in R^n locally taking orbits of the nonlinear flow ϕ_t of (3.3.1), to those of the linear flow $e^{tDf(\bar{x})}$ of (3.3.2). The homeomorphism preserves the sense of orbits and can also be chosen to preserve parameterization by time.

A more delicate situation in which the nonlinear and linear flows are related via diffeomorphisms (Sternberg's theorem) requires certain non-reaonance conditions among the eigenvalues of $Df(\bar{x})$. We shall not consider this here.

When $Df(\bar{x})$ has no eigenvalues with zero real part , $\bar{x}$ is called a hyperbolic or nondegenerate fixed point and the asymptotic behavior of solutions near it(and hence its stability type) is determined by the linearization. If any one of the eigenvalues has zero real part, then stability cannot be determined by linearization, as the example.

$$\ddot{x} - \varepsilon x^2 \dot{x} + x = 0 \qquad (3.3.4)$$

shows Rewritten as a system (with $x_1 = x$, $x_2 = \dot{x}$),

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \varepsilon \begin{pmatrix} 0 \\ x_1^2 x_2 \end{pmatrix} \qquad (3.3.5)$$

We find eigenvalues $\lambda, \bar{\lambda} = \pm i$. However, unless $\varepsilon = 0$, the fixed point $(x_1, x_2) = (0,0)$ is not a center, as in the linear system, but a non hyperbolic or weak attracting spiral sink if $\varepsilon > 0$, and a repelling source if $\varepsilon < 0$.

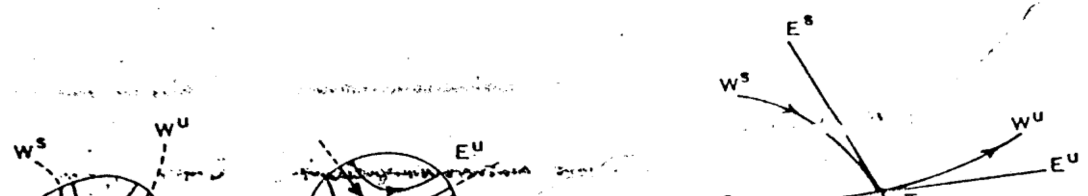
Before the next result we need a couple of definitions. We define the local stable and unstable manifolds of $\bar{x}$, $W^s_{loc}(\bar{x})$, $W^u_{loc}(\bar{x})$ as follows.

$$W^s_{loc}(\bar{x}) = \{x \in U / \phi_t(x) \rightarrow \bar{x} \text{ as } t \rightarrow \infty, \text{ and } \phi_t(x) \in U \text{ for all } t \geq 0\},$$
$$W^u_{loc}(\bar{x}) = \{x \in U / \phi_t(x) \rightarrow \bar{x} \text{ as } t \rightarrow -\infty, \text{ and } \phi_t(x) \in U \text{ for all } t \leq 0\}, \quad (3.3.6)$$

Where $U \subset R^n$ is a neighborhood of the fixed point x. The invariant manifolds W^s_{loc} and W^u_{loc} provide nonlinear analogues of the flat stable and unstable eigenspaces E^s, E^u of the linear problem(3.3.2). The next result tells us that W^s_{loc} and W^u_{loc} are in fact tangent to E^s, E^u at x.

Theorem 3.3.2. (Stable Manifold Theorem for a Fixed **point**). Suppose that $\dot{x} = f(x)$ has a hyperbolic fixed point $\bar{x}$. Then there exist local stable and unstable manifolds $W^s_{loc}(\bar{x})$, $W^u_{loc}(\bar{x})$ of the same dimensions n_s, n_u as those of the eigenspaces E^s, E^u the linearized system (3.3.2), and tangent to E^s, E^u at x. $W^s_{loc}(x)$, $W^u_{loc}(x)$ are as smooth as the function f.

For proofs of these two theorems see, for example, Hartman(1964) and Car (1980), or , for a more modern treatment, Nitecki (1971), Shub (1978), or Irwin (1981). Hirsch



et al, (1977) contains a more heneral result. The two result may be illustrated as in .Figure (2).

Figure (2) linearization and invariant subspaces.(a) Hartman's theorem ;(b) local stable and unstable manifolds

Note that we have not yet said anything about a center manifold, tangent to E^c at x, and have, in fact, confined ourselves to hyperbolic cases in which E^c does not exist.

The local invariant manifolds W^s_{loc} , W^u_{loc} have global analogues W^s, W^u, obtained by letting points in W^s_{loc} flow backwards in time and those in W^u_{loc} flow forwards:

$$W^s(\bar{x}) = \bigcup_{t\leq 0}\phi_t(W^s_{loc}(\bar{x})),$$
$$W^u(\bar{x}) = \bigcup_{t\geq 0}\phi_t(W^u_{loc}(\bar{x})). \qquad (3.3.7)$$

Existence and uniqueness of solutions of (3.3.1) ensure that two stable (or unstable) manifolds of distinct fixed points $\bar{x}^1$, $\bar{x}^2$ cannot intersect, nor can $W^s(\bar{x})$ (or $W^u(\bar{x})$) intersect itself. However, intersections of stable and unstable manifold of distinct fixed points or the same fixed point can occur and, in fact, are a source of much of the complex behavior found in dynamical systems. The global stable and unstable manifolds need not be embedded submanifolds of R^n since they may wind around in a complex manner, approaching themselves arbitrily closely. We give an example of a map poscssing such a structure in the next section.

To illustrate the ideas of this section, we consider a simple system on the plane:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -y + x^2, \end{aligned} \tag{3.3.8}$$

which has a unique fixed point at the origin. For the linearized system we have the following invariant subspaces:

$$\begin{aligned} E^s &= \{(x,y) \in R^2 / x = 0\}, \\ E^u &= \{(x,y) \in R^2 / y = 0\}, \end{aligned} \tag{3.3.9}$$

In this case we can integrate the nonlinear system exactly. Rather than obtaining a solution in the form(x(t), y(t)), we rewrite (3.3.8) as a(linear)first-order system by eliminating time:

$$\frac{dy}{dx} = \frac{-y}{x} + x \tag{3.3.10}$$

This can be integrated directly to obtain the family of solution curves

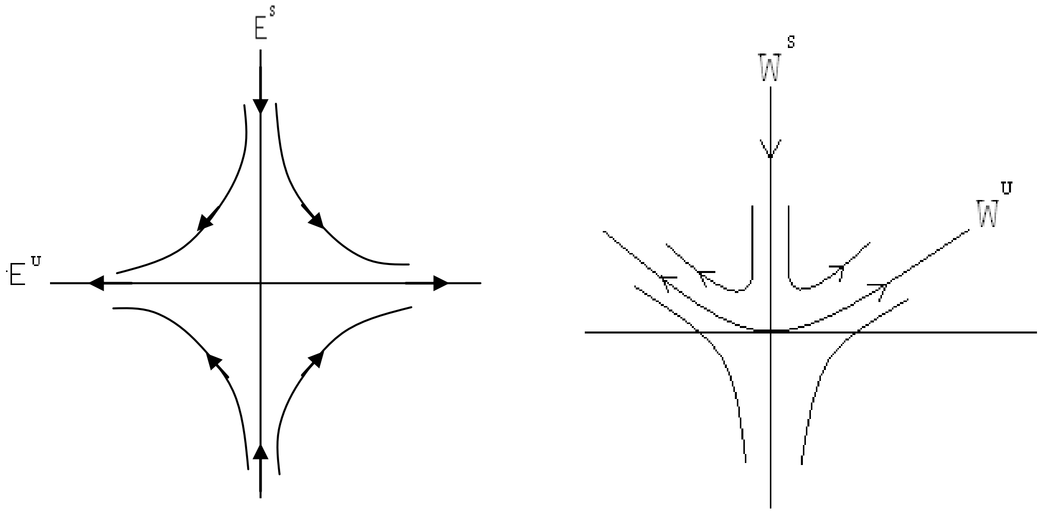
$$y(x) = \frac{x^2}{3} + \frac{c}{x}, \tag{3.3.11}$$

Where c is constant determined by initial conditions. Now Theorem 3.3.1, together with (3.3.9), implies that $W^u_{loc}(0,0)$ can be represented as a graph y = h(x) with h(0) = h(0) = 0, since W^u_{loc} is tangent to E^u at (0,0). Thus c = 0 in (3.3.10) and we have

$$W^u(0,0) = \left\{(x,y) \in R^2 / y = x^2\right\} \tag{3.3.12}$$

Finally, noting that if x (0) = 0, then x ≡ 0, and hence x (t) ≡ 0, we see that $W^s(0,0) \equiv E^s$. Note that , for this example, we have found the global manifolds; see Figure (3).

It is well known that nonlinear systems possess limit sets other than fixed points; for example, closed or periodic orbits frequently occur. A periodic solution is one for which there exists $0 < T < \infty$ such that x (t) = x(t+T) for all t. We consider the stability of such orbits in section 3.5, but note here that they have stable and unstable manifolds just as do fixed points.



(a) (b)

Figure(3) Stable and unstable manifolds for equation (3.3.8).(a) the linear system; (b) the nonlinear system.

Let γ denote the closed orbit and U be some neighborhood of γ; then we define.

$$W^s_{loc}(\gamma) = \{ x \in U \mid |\phi_t(x) - \gamma| \mapsto 0 \text{ as } t \mapsto \infty, \text{and } \phi_t(x) \in U \text{ for } t \geq 0 \}$$

$$W^u_{loc}(\gamma) = \{ x \in U \mid |\phi_t(x) - \gamma| \mapsto 0 \text{ as } t \mapsto \infty, \text{and } \phi_t(x) \in U \text{ for } t \leq 0 \}$$

3.4. Linear and Nonlinear Maps:

We have seen how the linear system (3.4.1) gives rise to flow map $e^{tA}: R^n \rightarrow R^n$, when e^{tA} is an n x n matrix. For fixed t = τ let e^{tA} = B, then B is a constant coefficient matrix and the difference equation.

$$x_{n+1} = Bx_n \quad \text{or} \quad x \rightarrow Bx, \qquad (3.4.1)$$

Is a discrete dynamical system obtained from the flow of (3.1.1). Similarly, a nonlinear system and its flow ϕ_t give rise to a nonlinear map.

$$x_{n+1} = G(x_n) \quad \text{or} \quad x \rightarrow G(x), \qquad (3.4.2)$$

Where G = ϕ_t is a nonlinear vector valued function. If the flow ϕ_t is smooth (say r-times continuosly differentiable), then G is a smooth map with a smooth inverse: i.e., a deffeomorphism. This is one example of the way in which a continuous flow gives rise to a discrete map; a more important one, the Poincare map will be considered in Section 3.5.

Deffeomorphisms or discrete dynamical systems can also be studied in their own right and more generally we might also consider noninvertible maps such as

$$x \rightarrow x - x^2 \qquad (3.4.3)$$

An orbit of a linear map $x \rightarrow Bx$ is a sequence of points $\{x_i\}_{i=-\infty}^{\infty}$ defined by $x_{i+1} = Bx_i$. Any initial point generates a unique orbit provided that B has no zero eigenvalues.

We define stable, unstable, and center subspaces in a manner analogous to that for linear vector fields:

E^s = span $\{n_s$ (generalized) eigenvector

whose eigenvalues have modulus < 1$\}$,

E^u = span $\{n_u$ (generalized) eigenvector

whose eigenvalues have modulus > 1$\}$,

E^c = span $\{n_c$ (generalized) eigenvector

whose eigenvalues have modulus = 1$\}$,

Where the orbits E^s and E^u are characterized by contraction and expansion, respectively. If there are no multiple eigenvalues, then the contraction and expansion are bounded by geometric series: i.e., there exist constants c > 0, α < 1 such that , for n ≥ 0,

$$
\begin{aligned}
|x_n| &\le c\,\alpha^n |x_o| \text{ if } x_o \in E^s, \\
|x_{-n}| &\le c\,\alpha^n |x_o| \text{ if } x_o \in E^u,
\end{aligned}
\qquad (3.4.4)
$$

If multiple eigenvalues occur, then much as in the case of flows, the contraction (or expansion) need not be exponential, However, an exponential bound can still be found if $|\lambda j| < 1$ for all eigenvalues.

In spite of problems caused by multiplicities, if B has no eigenvalues of unit modulus, the eigenvalues alone serve to determine stability. In this case x = 0 is called a hyperolic fixed point and in general, if $\bar{x}$ is a fixed point for G ($G(\bar{x}) = \bar{x}$ and $DG(\bar{x})$ has no eigenvalues of unit, modulus, then $\bar{x}$ is called a hyperbolic fixed point.

There is a theory for diffeomorphisms parallel to that for flows , and in particular the linearization theorem of Hartman – Grobman and the invariant manifold results apply to maps just as the flows (Hartman (1964) Nitecki (1971) , Shub(1978):

Theorem 3.4.1. (Hartman – Gobman). Let G: $R^n \longrightarrow R^n$ be a (C^1) diffeomorphism with a hyperbolitc fixed point $\bar{x}$. Then there exists a homeomorphism h defined on some neighborhood U on $\bar{x}$ such that h (G(ζ)) = DG(x) h (ζ) for all $\zeta \in U$.

Theorem 3.4.2. (Stable Manifold Theorem for a Fixed Point). Let G: $R^n \to R^n$ be a (C^1) diffeomorphism with a hyperbolic fixed point $\bar{x}$. Then there are local stable and unstable manifolds $W^s_{loc}(\bar{x})$, $W^u_{loc}(\bar{x})$, tangent to the eigenspaces $E^s_{\bar{x}}$, $E^u_{\bar{x}}$ of $DG(\bar{x})$ at $\bar{x}$ and of corresponding dimensions. $W^s_{loc}(\bar{x})$, $W^u_{loc}(\bar{x})$, are as smooth as map G.

Global stable and unstable manifolds are defined as for flows, by taking unions of backward and forward iterates of the local manifolds. We have

$W^s_{loc}(\bar{x}) = \left\{x \in U \,\middle|\, G^n(\bar{x}) \longrightarrow \bar{x} \text{ as } n \longrightarrow +\infty, \text{ and } G^n(x) \in U, \forall n \geq 0 \right\}$,

$W^u_{loc}(\bar{x}) = \left\{x \in U \,\middle|\, G^{-n}(\bar{x}) \longrightarrow \bar{x} \text{ as } n \longrightarrow +\infty, \text{ and } G^{-n}(x) \in U, \forall n \geq 0 \right\}$,

and

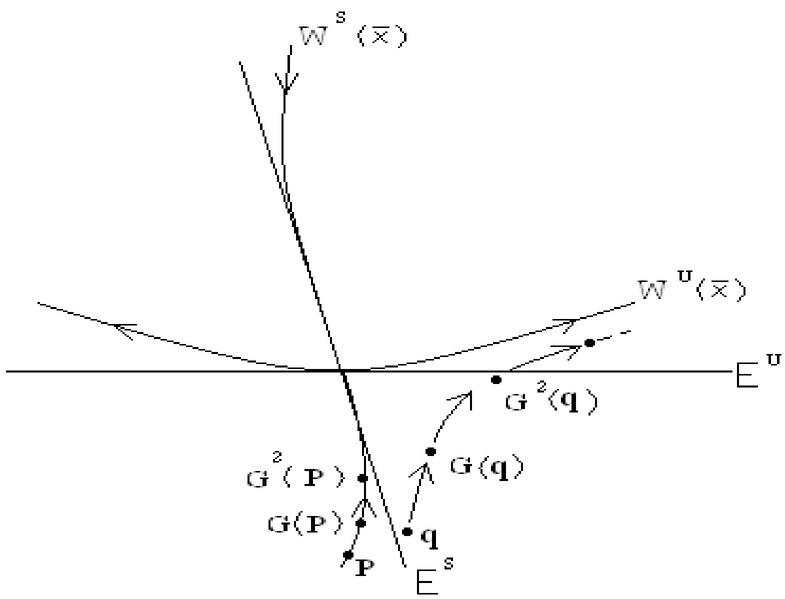
$$W^s(\bar{x}) = \bigcup_{n\geq 0} G^{-n}(W^s_{loc}(\bar{x})),$$

$$W^u(\bar{x}) = \bigcup_{n\geq 0} G^{n}(W^u_{loc}(\bar{x})),$$

We should bear in mind, however, that flows and maps differ crucially in mind, however, that flows and maps differ crucially in that, while the orbit or trajectory $\phi_t(p)$ of a flow is a curve in R^n, the

orbit $\{G^n(p)\}$ of a map is a sequence of points, Thus, while the invariant manifolds of flows are composed of the unions of solution curves. Those of maps are unions of discrete orbit points, see Figure(4) This distinction will be important later, in the discussion of global behavior.

We note that, when we write $G^2(p)$, we mean $G(G(p))$ and, similarly, that $G^n(p)$means the nth iterate of p under G. Thus, if there is a cycle of k distinct points $p_j = G^j(p_o)$, $j = 0 \ldots, k - 1$, and $G^k(p_o) = p_o$, we have a periodic orbit of period k. The stability of such an orbit is determined by the linearize map $DG^k(p_o)$, or, equivalently $DG^k(p_j)$ for any j. By the chain rule, we have $DG^k(p_o) = DG(G^{k-1}(p_o)) \ldots DG(G(p_o)). DG(p_o)$.



Figure(4) Invariant manifolds and orbits for a map $G:R^2 \longrightarrow R^2$

Much as for flows, the behavior of the linear map (3.4.1) is governed by the eigenvalues and the eigenvectors of B. Since maps are rarely dealt with in texts on differential equations or nonlinear oscillations, we include some details here. For a one-dimensional map, where B = b is a scalar and the orbit of a point $\{p_j\}_{j=0}^{\infty}$ is simply given by the geometric sequence $p_j = b^j p_o$, there are four "common" cases and three "unusual" ones listed below in Table (1).

Table (3.4.1) Behavior of the Linear Map $x \to bx$

Case	Description	Sketch
1. $b < -1$	Orientation reversing source	P_1 0 P_0 P_2
2. $b \in (-1, 0)$	Orientation reversing sink	P_1 0 P_2 p_o
3. $b \in (0,1)$	Orientation preserving sink	0 P_2 P_1 P_0
4. $b > 1$	Orientation preserving source	0 P_0 P_1 P_1
5. $b = -1$	Orientation reversing, all points of period 2	P_1 0 $P_0 = P_2$
6. $b = 0$	All points go to 0 on first iterate(noninvertible)	$0 = p_j, j \geq 1$ p_o
7. $b = +1$	Orientation preserving, all points fixed	0 $po = pj, \forall j$

In general, the stability type of the fixed point x = 0 is determined by the magnitude of the eigenvalues of B. If $\lambda_j < 1$ for all eigenvalies, then we have a sink ; if $\lambda_j > 1$ for some eigenvalues: a source. If $\lambda_j = 1$ for any eigenvalues then a norm is preserved in the directions v^j associated with those eigenvalues (unless they are multiple with nontrivial Jordan blocks).

If an even number of eigenvalues have negative real parts, then the map $x \to Bx$ is orientation preserving, while if an odd number have negative real parts it reverses orientation. We give some two-dimensional examples in Figure (5).

To get a feel for the rich and complex behavior possible for nonlinear maps we may like to experiment with the following two examples. Solutions may be conveniently obtained on a programmable pocket calculator or a minicomputer:

As a final example of a two-dimensional map with rather rich behavior, consider the simple linear map.

$$\begin{pmatrix} x \\ \ \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ & \end{pmatrix} \begin{pmatrix} x \\ \ \end{pmatrix}, \quad (x,y) \in T^2 = R^2/Z^2, \qquad (3.4.5)$$

y 1 2 y

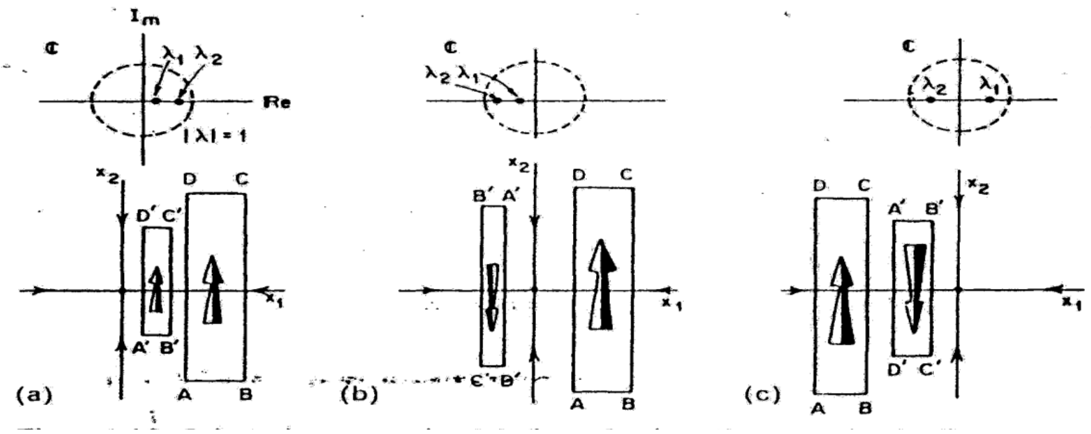


Figure (5) orientation preserving (a), (b) and orientation reversing (c)linear maps

Position of eigenvalues with respest to unit circle in complex plane shown above orbit structures. The oriented rectangle ABCD is mapped to A′B′C′ D′ in each case.

$$x \rightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x.$$

Where the phase space is the two-dimensional torus. On the plane (the covering space) we simply have a saddle point,

with eigenvectors $v^{1,2} = (1,(1\pm\sqrt{5})/2)^T$ belonging to the eigenvalues $\lambda_{1.2} = (3 \pm\ \ 5)/2$. Since the map is linear , $W^s(0) = E^s$, $W^u(0) = E^u$ and thus span $\{(1,(1+\sqrt{5})/2)^T\}$ is the unstable manifold and span $\{(1,(1-\sqrt{5})/2)^T\}$the stable manifold. However, Our phase space is the torus, T^2, obtained by identifying points whose coordinates differ by integers. The map is well defined on T^2 since it preserves the periodic lattice. Any point of the unit square 0,1) × 0 , 1) mapped into another square is translated back into the original square; for example, if (x, y) =

(- 1.4, + 1.2), we set (x, y) = (0.6 , 0.2). See Figure (6). Thus the unstable manifold " runs off the square" at (2/(1+ 5), 1) and reappears, with the same slope , at (2/(1+ 5),0) to run off at (1, (5 -1)/2), etc. Since the slopes of W^s and W^u are irrational ((1 ± 5)/2) these manifolds are dense in the unit square (or wind densely around the torus). Thus each manifold approaches itself arbitrarily closely, and hence is not an embedded submanifold of T^2. Arnold and Avez 1968, pp.5-7 have nice illustrations of the torus map.

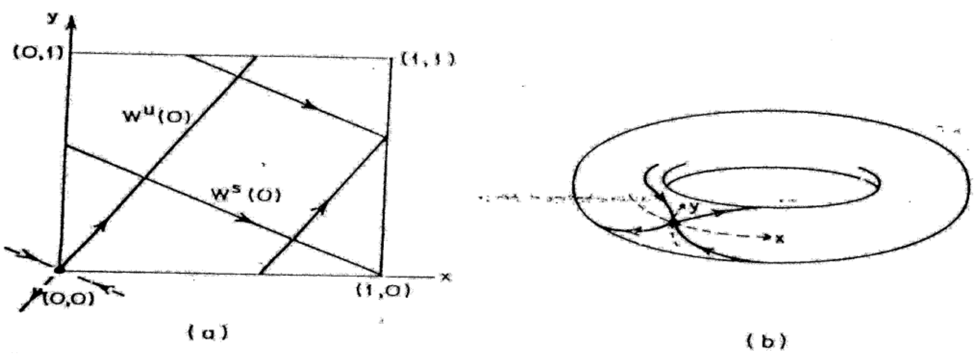


Figure (6).The linear map on the torus(the hyperbolic toral autmorphism)(a) On $R^2$the covering space;(b) on T^2.

3.5.Closed Orbits, Poincare Maps, and Forced Oscillations:

In classical texts on differential equations the stability of closed orbits or periodic solutions is discussed in terms of the characteristic or Floquet multipliers. Here we wish to introduce a more geometrical view which is in essence equivalent: the Poincare map. Since the ideas are so important, we devote a considerable amount of space to familiar examples from forced oscillations.

Let γ be a periodic orbit of some flow ϕ_t in R^n arising from a nonlinear vector field $f(x)$. We first take a local cross section $\Sigma \subset R''$, of dimension n – 1. The hypersurface Σ need not be planar, but must be chosen so that the flow is everywhere transverse to it. This is achieved if $f(x) . n(x) \neq 0$ for all $x \in \Sigma$, where n(x) is the unit normal to Σ at x. Denote the (unique) point where γ intersects Σ by p, and let $U \subseteq \Sigma$ be some neighborhood of p.(If γ has multiple intersections with Σ, then shrink Σ until there is only one intersection).Then the first return or Poincare map $P:U \rightarrow \Sigma$ is defined for a point $q \in U$ by.

$$P(q) = \phi_\iota(q), \qquad (3.5.1)$$

Were $\iota = \iota(q)$ is the time taken for the orbit $\phi_t(q)$ based at q to first return to Σ. Note that ι generally depends upon q and need not be equal to T = T(p), the period of γ, However, $\iota \rightarrow T$ as $q \rightarrow p$.

Clearly p is a fixed point for map P , and it is not difficult to see that the stability of p for P reflects the stability of γ for the flow ϕ_t. In particular, if p is hyperbolic, and DP(p), the linearized map, has n_s eigenvalues with modulus less than one and n_u with modulus greater than one ($n_s + n_u = n - 1$), then dim W^s (p) = n_s, and dim $W^u(p) = n_u$ for the map. Since the orbits of P lying in W^s and W^u are formed by intersestions of orbits(solution curves) of ϕ_t with Σ, the dimensions of $W^s(\gamma)$ and

$W^u(\gamma)$ are each one greater than those for the map [5]. This is most easily seen in the sketches of Figure (7).

As an example, consider the planar system.

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2), \\ \dot{y} &= x + y - y(x^2 + y^2), \end{aligned} \qquad 3.5.2$$

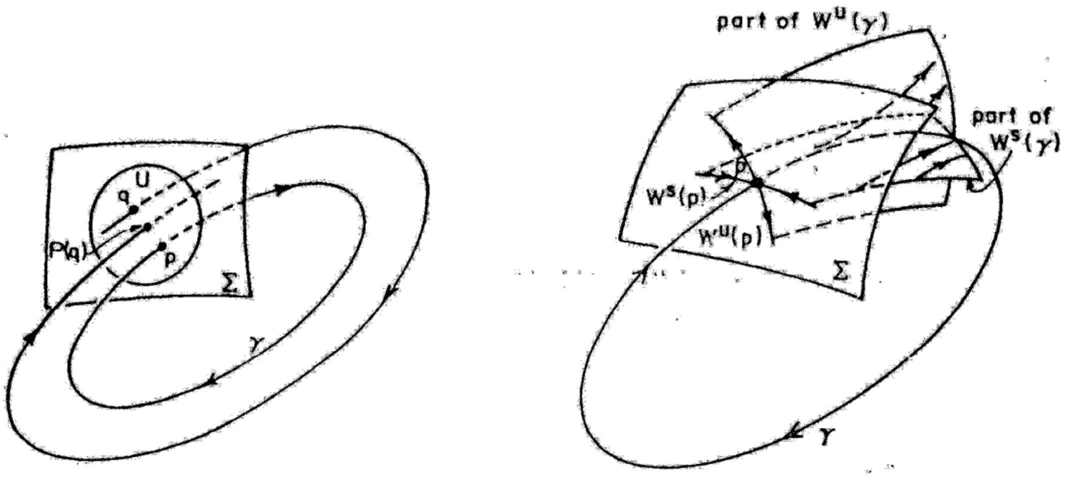


Figure (7) The Poincare map. (a) The cross section and the map; (b) a closed orbit.

and take our cross section

$$\Sigma = \{(x, y) \in R^2 \mid x > 0, y = 0\}.$$

Transforming (3.5.2) to polar coordinates $r = (x^2 + y^2)^{1/2}$, $\theta = \arctan(y/x)$, we obtain.

$$\begin{aligned} \dot{r} &= r(1 - r^2), \\ \dot{\theta} &= 1, \end{aligned} \qquad (3.5.3)$$

and the section becomes

$$\Sigma = \{(r,\theta) \in R^+ \times S^1 \mid r > 0, \theta = 0\}.$$

It is easy to solve (3.5.3) to obtain the global flow

$$\phi_t(r_0, \theta_0) = ((1 + (1 - 1)e^{-2t})^{-1/2}, t + \theta_0)$$

$\overline{\quad} r^2_0$

The time of flight ι for any point $q \in \Sigma$ is simply $\iota = 2\pi$, and thus the Poincare map is given by

$$P(r_o) = (1+(\frac{1}{r^2_o} - 1)e^{-4\pi})^{-1/2} \qquad (3.5.4)$$

Clearly, P has a fixed point at $r_o = 1$, reflecting the circular closed orbit of radius 1 of (3.5.3). Here P is a one-dimensional map and its linearization is given by.

$$DP(1) = \left.\frac{dP}{dr_o}\right|_{r_o=1} = -\frac{1}{2}(1+(\frac{1}{r^2_o}-1)e^{-4\pi})^{-3/2} \left.\frac{2e^{-4\pi}}{r^3_o}\right|_{r_o=1}$$

$$= e^{-4\pi} < 1. \qquad (3.5.5)$$

Thus $p = 1$ is a stable fixed point and γ is a stable or attracting closed orbit[5] .

We note that we could have computed DP(1) a little more simply by considering the flow of the vector field (3.5.3) linearized near the closed orbit r =1. Since $(d/dr)(r - r^3) = 1 - 3r^2$, this is

$$\xi = 2\xi, \qquad \dot{\theta} = 1, \qquad (3.5.6)$$

with flow

$$D\phi_t(\xi_0,\theta_0) = (e^{-2t}\xi_0, t + \theta_0). \qquad (3.5.7)$$

Hence $DP(1) = e^{-2(2\pi)} = e^{-4\pi}$, as above

To demonstrate the general relationship between Poincare maps and linearized flows we must review a little Floquet theory (Hartman (1964), §§IV.6,IX.10).Let $\overline{x}(t) = \overline{x}(t + T)$ be a solution lying on the closed orbit γ, based at $x(0) = p \in \Sigma$.

Linearizing the differential equation about γ, we obtain the system

$$\xi = Df(\bar{x})(t))\xi, \tag{3.5.8}$$

Where $Df(\bar{x}(t))$ is an $n \times n$, T-periodic matrix. It can be shown that any fundamental solution matrix of such a T-periodic system can be written in the form

$$X(t) = Z(t)e^{tR}; \qquad Z(t) = Z(t + T), \tag{3.5.9}$$

Where X, Z, and R are $n \times n$ matrices (cf. Hartman (1964), P.60).In particular, we can choose $X(0) = Z(0) = 1$, so that

$$X(T) = Z(T)e^{TR} = Z(0)e^{TR} = e^{TR} \tag{3.5.10}$$

It then follows that the behavior of solutions in the neighborhood of γ is determined by the eigenvalues of the constant matrix e^{TR}. These eigenvalues, $\lambda_1, \ldots, \lambda n$, are called the characteristic(Floquet)multipliers or roots and the eigenvalues $\mu_1, \ldots, \mu_n$ of R are the characteristic exponents of the closed orbit γ. The multiplier associated with perturbations along γ is always unity; let this be λ_n. The moduli of the remaining $n - 1$, if none are unity, determine the stability of γ.

Choosing the basis appropriately, so that last column of e^{TR} is $(0,\ldots, 0,1)^T$, the matrix DP(p) of the linearized Poincare map is simply the $(n - 1) \times (n-1)$ matrix obtained by deleting the nth row and column of e^{TR}. Then the first n -1 multipliers λ_1, …,λ_{n-1} are the eigenvalues of the Poincare map.

Although the matrix R in (3.5.9) is not determined uniquely by the solutions of (3.5.8) (Hartman (1964), P.60), the eigenvalues of e^{TR} are uniquely determined (e^{TR} can be replaced by any similar matrix $C^{-1}e^{TR}C$).However, to compute these eigenvlues we still need a representation of e^{TR} and this can only be obtained by actually generating a set of n linearly independent solutions to form X(t). Except in special cases, like the simple example above, this is generally difficult and requires perturbation methods or the use of special functions.

We have seen how a vector field $f(x)$ on R^n gives rise to a flow map ϕ_t on R^n and, in the neighborhood of a closed orbit, to a (local) poincare map P on a transversal hypersurface Σ. Another important way in which a flow gives rise to a map is in non-autonomous, periodically forced oscillations. Consider a system.

$$x = f(x, t); \qquad (x, t) \in R^n \times R, \qquad (3.5.11)$$

Where $f(.,t) = f(.,t + T)$ is periodic of period T in t. System (3.5.11) may be rewritten as autonomous system at the expense of an increase in dimension by one, if time is include as an explicit state variable:

$$\begin{aligned} \dot{x} &= f(x, \theta), \\ \dot{\theta} &= 1; \quad (x, \theta) \in R^n \times S^1 \end{aligned} \qquad (3.5.12)$$

The phase space is the manifold R^n x S^1, where the circular component S^1 = R (mod T) reflects the periodicity of the vector field f in θ. For this problem we can define a global cross section.

$$\Sigma = \left\{(x, \theta) \in R^n \times S^1 \mid \theta = \theta_0\right\} \quad , \qquad (3.5.13)$$

Since all solutions cross Σ transversely, in view of the component $\dot{\theta}$ = 1 of (3.5.12). The Poincare map P: Σ → Σ, if it is defined globally, is given by

$$P(x_0) = \pi .\ \phi_T(x_0,\theta_0) \qquad (3.5.14)$$

Where $\phi_t: R^n \times S^1 \to R^n \times S^1$ is the flow of (3.5.12) and π denotes projection onto the first factor. Note that here the time of flight T is the same for all points x ∈Σ. Alternatively, $P(x_o)$ = x $(x_o, T + \theta_o)$, where $x(x_o, t)$ is the solution of (3.5.12) based at x $(x_o, \theta_o) = x_o$.

The Poincare map can also be derived as a discrete dynamical system arising from the flow ψ(x, t) of the time-dependent vector field of (3.5.11). Since f is T-periodic, we have $\psi\ (x, nT) \equiv \psi^n(x, T) = \psi^n{}_T(x)$. The map $P(x_o) = \psi_T(x_o)$ is in this sense another example of a discrete dynamical system [5].

The system:

$$\dot{x} = x^2, \qquad (3.5.15)$$

$$\dot{\theta} = 1,$$

with solution

$$\phi_t(x_o, \theta_o) = \left(\left(\frac{1}{x_o} - t\right)^{-1}, t + \theta_o\right),$$

and the Poincare map

$$P(x_0) = \left(\frac{1}{x_0} - 2\pi\right)^{-1}, \; x_0 \in (-\infty, 1/2\pi)$$

on $\Sigma = \{(x, \theta) \mid \theta = 0\}$ shows that P may not be globally defined. Here , trajectories of ϕ_t based at $x_o \geq 1/2\pi$ approach ∞ at a time $t \leq 2\pi$. However, $P: U \rightarrow \Sigma$ is usually defined for some subset $U \subset \Sigma$.

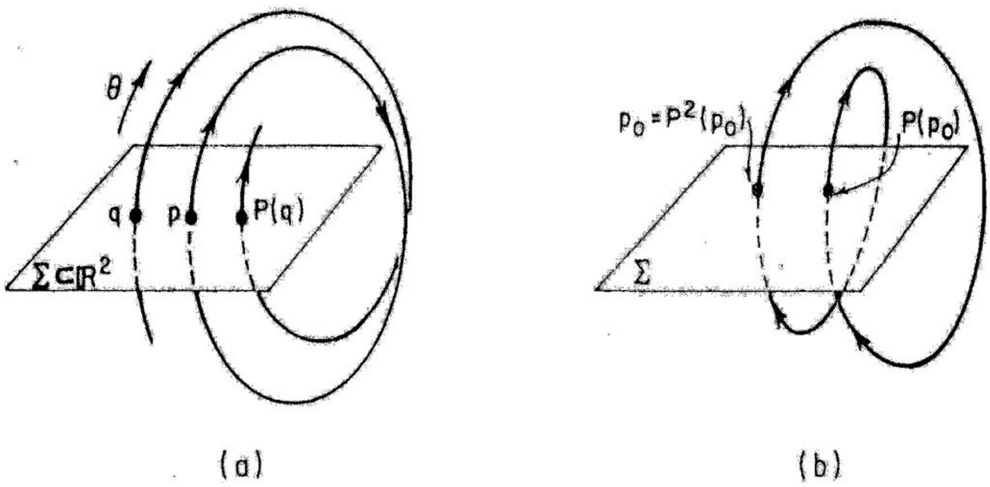


Figure ٨ The Poincaré map for forced oscillations. (a) A periodic orbit of period T and the fixed point $p = P(p)$; (b) a subharmonic of period $2T$.

We illustrate the Poincare map for forced oscillations in Figure (8). As in the previous case, it is easy to see that a fixed point p of P corresponds to a periodic orbit of period T for the flow. In addition, a periodic point of period $k > 1(P^k(p) = p$ but $P^j(p) \neq p$ for $1 \leq j \leq k - 1)$ corresponds to a subharmonic of period kT. Here P^k means P iterated k times, thus $P^2(p_o) = P(P(p_o)$; etc. This, of course, also applies for the autonomous case discussed earlier. Such periodic points must always come in sets of : $p_o, \ldots, p_{k-1}$ such that $P(p_i) = p_{i+1}$, $0 \leq i \leq k - 2$ and $p_o = P(p_{k-1})$.

Forced Linear Oscillations:

We start with a problem for which a general solution can be found and the Poincare map computed explicitly. Consider the system.

$$\ddot{x} + 2\beta\dot{x} + x = \gamma \cos \omega t; \qquad 0 \leq \beta < 1, \qquad (3.5.16)$$

Or

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma \cos \omega\theta \end{pmatrix}, \qquad (3.5.17)$$

$$\dot{\theta} = 1.$$

Here the forcing is of period $T = 2\pi/\omega$. Since the system is linear, its solution is easily obtained by conventional methods (cf. Braun 1978 :

$$x(t) = e^{-t\beta}(c_1 \cos \omega_d t + c_2 \sin \omega_d t) + A \cos \omega t + B \sin \omega t, \qquad (3.5.18)$$

where $\omega_d = \sqrt{1 - \beta^2}$ is the damped natural frequency and A and B, the coeffic-ients in the particular solution, are given as

$$A = \frac{\quad}{(1 - \omega^2)\gamma}, \qquad B = \frac{\quad}{2\beta\omega\gamma} \qquad (3.5.19)$$

$(1-\omega^2)^2 + 4\beta^2\omega^2)$ $(1-\omega^2)^2 + 4\beta^2\omega^2)$

The constants c_1, c_2 are determined by the initial conditions. Letting $x = x_1 = x_{10}$ and $x = x_2 = x_{20}$, at $t = 0$, we have

$$\left.\begin{array}{l} x(0) = x_{10} = c_1 + A \\ x(0) = x_{20} = -\beta c_1 + \omega_d c_2 + \omega B \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = x_{10} - A, \\ c_2 = (x_{20} + \beta(x_{10} - A) - \omega B)/\ \omega_d \end{array} \quad (3.5.20)$$

Thus, since $\phi_t(x_{10}, x_{20}, 0)$ is given by (3.5.18) and

$$\begin{aligned} x_2(t) = x_1(t)\, e^{-\beta t} \ -\ \beta(c_1 \cos \omega_d t + c_2 \sin \omega_d t) \\ + \omega_d(-c_1 \sin \omega_d t + c_2 \cos \omega_d t) \\ - \ \omega(A \sin \omega t - B \cos \omega t), \end{aligned}$$

we can compute the Poincare map explicitly as $\pi.\ \phi_{2\pi/\omega}(x_{10},x_{20},0)$. In the case of resonance, $\omega = \omega_d = \sqrt{1 - \beta^2}$, we obtain

$$P(x_{10}, x_{20}) = ((x_{10} - A)e^{-2\pi\beta/\omega} + A,\ (x_{20} - \omega B)e^{-2\pi\beta/\omega} + \omega B. \quad (3.5.21)$$

As expected, the map has an attracting fixed point given by $(x_1, x_2) = (A, \omega B)$ or $c_1 = c_2 = 0$. The map is, of course, linear and since the matrix

$$\begin{pmatrix} \dfrac{\partial P_1}{\partial x_{10}} & \dfrac{\partial P_1}{\partial x_{20}} \\ \dfrac{\partial P_2}{\partial x_{10}} & \dfrac{\partial P_2}{\partial x_{20}} \end{pmatrix} = \begin{pmatrix} e^{-2\pi\beta/\omega} & 0 \\ 0 & e^{-2\pi\beta/\omega} \end{pmatrix} \quad (3.5.22)$$

is diagonal with equal eigenvalues, the orbits of P approach(A,ωB)radially, cf. Figure (9).

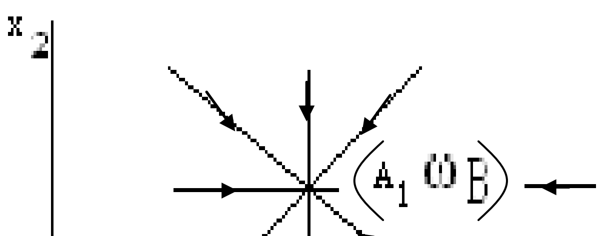


Figure (9) The poncar map of the linear oscillator equation.

Chapter four

The Stability of Bifurcations, Co dimensions of Equibilria and Periodic Orbits

In this chapter, we study the local bifurcation of vector fields and maps. As we have seen, systems of physical interest typically have parameters are which appear in the defining systems of equations. As these parameters are varied, changes may occur in the qualitative structure of the solution for certain parameter values. These changes are called bifurcation and the parameter values are called bifurcation values. To the extent possible, we develop in this chapter.

We start by considering some simple examples of bifurcations of fixed points of flows in one and two dimensions and go on to develop the general theory for dealing with bifurcations of fixed points of n-dimensional flows.

The principal components of this theory are the center manifold and normal form theorem. At the end of the chapter we turn our attention to local bifurcations of maps and develop an analogous theory for them.

4.1 Bifurcation Problems

The term bifurcation was originally used by Poincar'e to describe the "splitting" of equilibrium solutions a family of differential equations. If

$$\dot{x} = f_\mu(x), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^k \qquad (4.1.1)$$

is a system of differential equations dependi . dimensional parameter μ, then the equilibrium solutions of (4.1.1) are given by the solutions of the equation $f_\mu(x)=0$. As μ varies, the implicit function theorem implies that these equilibria are described by smooth functions of μ away from those point at which the jacobian derivative of $f_\mu(x)$ with respect to x, $D_x f_\mu$ has azero eigenvalue. The graph of each of these functions is a branch of equilibria of (4.1.1). At an equilibrium (x_o,μ_o) where $D_x f_\mu$, has a zero eigenvalue, several branches of equilibria may come together, and one says that (x_o,μ_o) is a point of *bifurcation.*

As an example consider equation (4.1.1)with $f_\mu(x) = \mu x - x^3$. Here $D_x f_\mu = \mu - 3x^2$,and the only bifurcation point is $(x,\mu)=(0,0)$.It is easy to check that the unique fixed point $x = 0$ existing for $\mu \leq 0$ is stable, that it becomes unstable for $\mu > 0$, and that the new bifurcating fixed points at $x = \pm\sqrt{\mu}$ are stable.

We obtain the qualitative picture of Figure (4.1.1) in which the branches of equilibria are shown in (x,μ) space. This figure is an example of a *bifurcation diagram* [5].

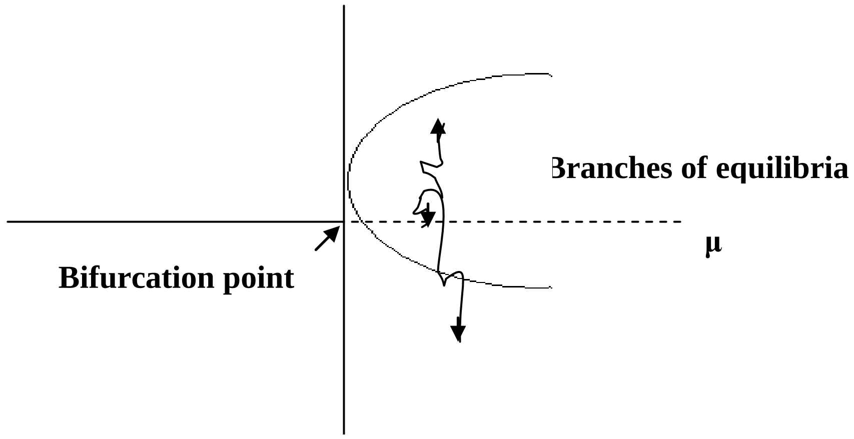


Figure (1). The bifurcation diagram for $f\mu = \mu x - x^3$ ---- sources; ---- sinks

In each case find the nontrivial fixed points and investigate their stability. Also sketch the bifurcation diagrams. Some of these bifurcations are more degenerate than others, in that small perturbations to f_μ can change the topological structure of their bifurcation diagrams. Can you identify these, and sketch some perturbed bifurcation diagrams?

Bifurcations of equilibria usually produce changes in the topological type of a flow, but there are many other kinds of changes that occur in the topological equivalence class of flows. We shall include all of these in our use of the term bifurcation.

Definition 4.1.1.

A value μ_0 of equation (4.1.1)for which the flow of (4.1.1) is not structurally stable is a bifurcation value of μ.

This definition is not completely satisfactory because it impels one to study the finely detailed structure of flows for which complete descriptions do not exist. Consequently, attempts to construct a systematic bifurcation theory lead to very difficult technical questions, not all of which have relevance for applications of the theory. To avoid such complications, we frequently loosen the definition given above and examine only some of the qualitative features of a system of differential equation. We do not, however, retreat to the "static" problem of dealing only with the bifurcations of equilibria (cf. Sattinger [1973]).

Another peculiarity of this definition is that a point of bifurcation need not actually represent a change in the topological equivalence class of a flow. For example , the system $\dot{x} = -(\mu^2 x + x^3)$ has abifurcation value $\mu = 0$, but all of the flows in this family have a globally attracting equilibrium at $x = 0$. However, arbitrary perturbations (unfoldings) do give topologically distinct flows.

Given a system (4.1.1), we want to draw its bifurcation set. This consists of the loic in μ-space which correspond to systems for which structural stability breaks down in specific ways which we classify to the extent that we are able to do so. We also sometimes find it convenient to draw bifurcation diagrams: the loci in the (x, μ) product space of (parts of)the invariant set of (4.1.1). these invariant sets need not merely be fixed points, as in Figure(1): periodic orbits, for example, are often represented in terms of some measure ($|x|$) of their amplitude.[5]

What is of particular interest is that there are identifiable kinds of bifurcation which appear repeatedly in many problems. I ideally, we would like to have a classification of bifurcations which produced a specific list of possibilities for each example, starting with only general considerations such as the number of parameters in the problem, the dimension of the phase space and any symmetries or other special properties of the system (e.g. the forced Duffing equation and the Lorenz systems are volume contracting: this excludes a lot of types of behavior in these systems). Parts of such a classification have been developed, The classification schemes are based upon concepts which have their origin in the theory of transversality in differential topology. The transversality theorem implies that when two manifolds (surfaces) of dimension k and I meet in n-dimensional space, then in general, their intersection will be a manifold f dimension ($k+1 - n$). If $k + 1 < n$ then one dose not expect intersections to occur at all. For example tow-dimensional surfaces in 3-space generally intersect along curves, while tow curves in 3-space generally do not intersect. The meaning of in general is given in terms of function space topologies for the space of embeddings of 1- dimensional manifolds in-space. We only remark here that non-transversal intersections can be perturbed to transversal ones , but transversal intersections retain their topology under perturbation. A general position or transversal intersection of manifolds in n-dimensional space is one for which the tangent spaces and we shall give a fairly complte survey of what is known in this chapter of the intersecting manifolds span *n*-space. The dimension formula can be expressed also in terms of codimension. The codimension of an *I*-dimensional submanifold of *n*-space is(n - 1). Then the intersection of two submanifold Σ_1, Σ_2 generally satisfies (n - 1) + ($n - k$) = 2n – ($1+ k$)= n – ($1 + k – n$). Therefore the codimension of $\Sigma_1 \cap \Sigma_2$ is the sum of the codimensions of Σ_1 **and** Σ_2 if the intersection is transversal.

As an example, consider two curves in the plane, one of which is the *x*-axis, the other being the graph of a function *f*. The two curves intersect transversally at a point *x* if $f(x) = 0$ (the intersection condition) *and* $f'(x) \neq 0$ (transversally). We say that a transversal intersection of the curves simple zero. If *f* has only simple zeros, then small perturbations of *f* have the same number of zeros as *f*. *In* a family f_μ the simple zeros vary smooth functions of µ. (This statement is just the implicit function theorem.) Nonsimple zeros do not have these properties. For example, the family $f_\mu(x) = \mu + x^2$ has a nonsimple zero at (x,µ) = (0,0). For µ > 0, the functions f_μ have no zeros at all. Not however, that if one regards f_μ(x) = F(x,µ) as function of two variable, then it's graph intersects the (x,µ) coordinate plane transversally along the curve $\mu + x_2 = 0$. Thus while the bifurcation point (x,µ) = (0,0) corresponds to an unstable system, the bifurcation diagram corresponding to the *family of systems* is stable to small perturbation. We note that looss and Joseph [1981] make a strong distinction between such " turning point" or "fold" bifurcations and branching bifurcations such as that of Figure (1).

4.2 Center Manifolds:

In this section we begin our development of the techniques necessary for the analysis of bifurcation problems. We discuss and state the center manifold theorem. Which provides a means for systematically reducing the dimension of the state spaces which need to be considered when analyzing bifurcations of a given type. We use the Lorenz system and its bifurcation at p = 1 as an example which illustrates the role of center manifolds in bifurcation calculations. There are two analogous situations to consider: an equilibrium for a vector filed and fixed point for a diffeomorphism. The second case often arises from the Poincare return map of a periodic orbit of a flow.

Suppose that we have system of ordinary differential equations $\dot{x} = f(x)$ such that f(0) = 0 . If the linearization of f at the origin has no pure imaginary eigenvalues, then Hartman's theorem[5]states that the numbers of eigenvalues with positive and negative real parts determine the topological equivalence of the flow near 0. If there are eigenvalues with zero real parts, then the flow near the origin can be quite complicated.

In general the center manifold method isolates the complicated asymptotic behavior by locating an invariant manifold tangent to the subspace spanned by the (generalized) eigenspace of eigenvalues on the imaginary axis. There are technical difficulties here that are not present in the stable manifold theorem, however. These involve the nonuniqueness and the loss of smoothness in the invariant center manifold . Before stating the main result, we illustrate these issues with a pair of examples.

Our first example is due to Kelley [1967]. Consider the system

$$\dot{x} = x^2 \ ,$$
$$\dot{y} = - y \ , \qquad (4.2.1)$$

The solutions to this system have the from $x(t) = x_o/(1-t\,x_o)$ and $y\,(t) = y_o e^{-t}$. Eliminating *t*, we obtain solution curves which are graphs of the functions $y\,(x) = (y_o e^{-1}/^{xo})e^{1/x}$. *For* $x < 0$, all of these solution curves approach the origin in a way which is " flat": that is, all of their derivatives vanish at $x = O$. For x > o the only solution curve which approaches the origin (as $t \to - \infty$) is the x-axis. Thus the center manifold, tangent to the direction of the eigenvector belonging to 0 (the *x*-axis) is far from unique. We can obtain C^∞ center manifold by piecing together any solution curve in the left half plane with the positive half of the x-axis; cf. Figure (2). Note however, that the *only analytic* center manifold is the x-axis itself.

To explain the lack of smoothness in center manifolds, we first make a simple observation about the trajectories which approach a node. Consider the linear system

$$\dot{x} = ax,$$
$$\dot{y} = by,$$

With b > a > 0. Dividing these equations, we obtain.

$$\frac{dy}{dx} = \frac{by}{ax}$$

The solutions of equation (4.2.3) are easily seen to have form $y(x) = C\,|x|^{(a/b)}$. The graphs of the functions *y(x)* are the solution curves of (4.2.2). If we extend one these solution curves to the origin, then it fails to be infinitely differentiable if *b/a* is not an integer and $C \neq 0$. If $r < b/a < r+1$, then the extended curve will be C^r but not C^{r+1}. Even if *b/a* is an integer, the curve formed from the union of 0 and two solution curves to the right and left of 0 will only be *b/a*-1 times differentiable in general [5].

We now give an example which illustrates that a center manifold may be forced to contain curves that are patched together at a node like those we have just described. Consider the system

$$\dot{x} = \mu x - x^3,$$
$$\dot{y} = y,$$
$$\dot{\mu} = 0,$$

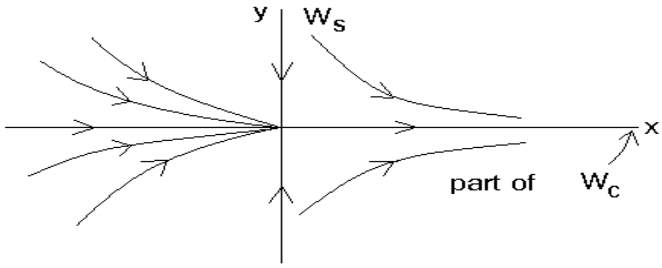


Figure (2). The phase portrait of equation (4.2.1), showing some center manifolds(heavy curves).

in which the "parameter" μ plays the role of a (trivial) dependent variable.

It is easy to verify that, for the system linearized at (x,y, μ)=(0.0.0), the y-axis is an unstable subspace and the (x, μ) plane the center subspace. The equilibria of this system consist of the μ -axis and the parabola $\mu = x^2$ in the (x, μ) coordinate plane, Figure (3) Since $\dot{\mu}$ = o, the planes μ = constant are invariant under the flow of (4.2.4). In a plane μ = constant ≠ o, all of the equilibria are hyperbolic. Those on the μ -axis with μ < 0 and along the parabola are saddles, while those along the positive μ-axis are unstable nodes. We want to find the center manifold of 0. For μ ≤ 0 the flow of (4.2.4.)is topologically a one parameter family of saddles, and the only choice for a center manifold comes from points in the (x, μ) coordinate plane.

When μ > 0 , the unstable manifolds of the saddles along the parabola (each a vertical line $x = \pm\sqrt{\mu}$) form an invariant manifold *M* which separates R^3 into two invariant regions. The center manifold must intersect M, but it can only do so by containing the parabola of equilipria. It follows that the center manifold for $\mu > 0$ must consist of the equilibria along the parabola. Of (4.2.4) together with stable saddle separatrices of the equilibria along the parabola. For (4.2.4) ,these all lie in the (x, μ) coordinate plane and the center manifold is the (x , μ) coordinate plane. However if we modify the system (4.2.4) by changing the second equation to $y = y + x^4$, then we assert (without proof) that the center manifold must still consist of the equilibria together with their saddle separatrices. However, these now no longer fit together in a C^∞ way along the curve of nodes on the positive half of the μ – axis. The degree of smoothness decreases as one moves away from the origin because the linearization of (4.2.4) at the point $(0,0, \mu_o)$ has eigenvalues in the plane $\mu = \mu_o$ which are 1 and μ_o . Therefore the degree of smoothness we expect is bounded by $1/ \mu_o$. If we are interested only in a C^r invariant mainiflod with $r < \infty$, then our search for one will be successful as long as we restrict attention to a sufficiently small neighborhood of the origin (of diameter at most 1 /r in this example).

With these examples as motivation, we now state

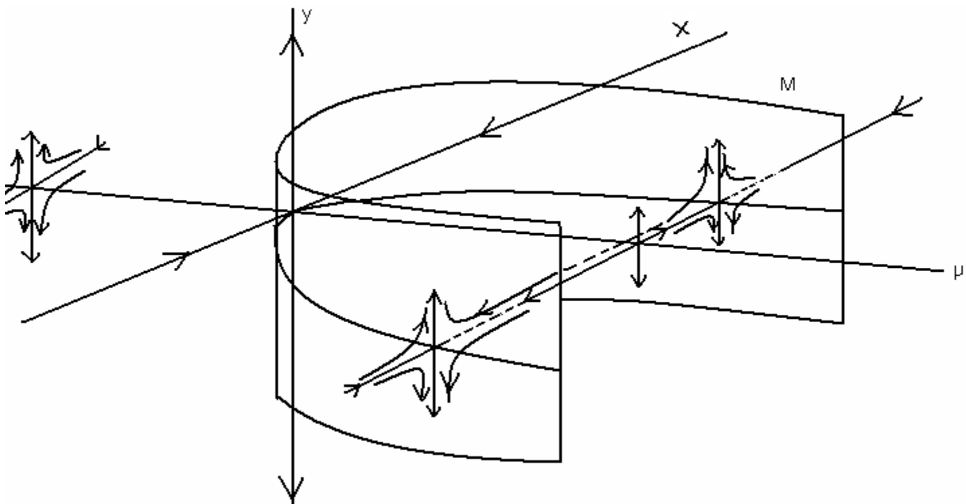


Figure (3). Invariant manifolds for equation (4.2.4.)

Theorem 4.2.1 (Center manifold Theorem for Flows).

Let *f* be a C^r vector filed on R^n vanishing at the origin ($f(0) = 0$)and let $A = Df(0)$, Divide the spectrum of A into three parts σ_s , σ_c , σ_u with

$$\operatorname{Re}\lambda \begin{cases} < 0 & \text{if } \lambda \in \sigma_s , \\ = 0 & \text{if } \lambda \in \sigma_c , \\ < 0 & \text{if } \lambda \in \sigma_u , \end{cases}$$

Let the (generalized) eigenspaces of σ_s , σ_c and σ_u be E^s, E^c and E^u respectively, then there exist C^r stable and unstable invariant manifolds W^u and W^s tangent to E^u and E^s at O and a C^{r-1} center manifold W^c tangent to E^c at O. the manifolds W^u, W^s, and W^c are all invariant for the flow of *f*. the stable and unstable manifolds are unigue, but W^c need not be.

We illustrate the situation in Figure (4) note that we cannot assign directions to the flow in W^c without specific information on the higher-order terms of *f* near o.

For more information on the existence, uniqueness, and smoothness of center manifolds and for profs of Theorem 3.2.1 and the results to follow, see Marsden and McCracken [1976], Carr [1981], and Sijbrand [1981] Kelley's [1967].

One might guess that a simpler alternative to using the center manifold theorem for a system would be to project the system onto the linear subspace spanned by E^c. Thus, if one writes a vector filed *as* $f = f_u + f_s + f_c$ with

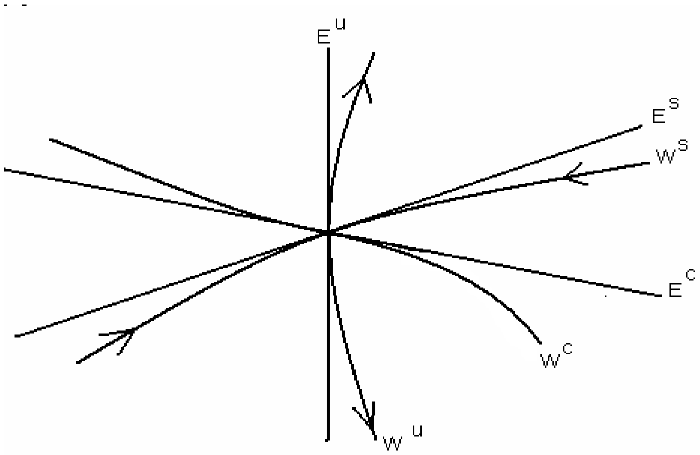


Figure (4)The stable, unstable, and center manifolds.

$f_u \ \varepsilon \ E^u$, $f_s \ \varepsilon \ E^s$, *and* $f_c \ \varepsilon \ E^c$, near the equilibrium one would hope that f_c restricted to E^c provides the correct qualitative picture of the dynamics in the center directions. The Lorenz system illustrates that this is not always the case and thus provides an instructive example of the role played by the center manifold calculations in a bifurcation problem[5].

Recall the Lorenz system

$$x = \sigma (y - x),$$
$$\dot{y} = px - y - xz, \qquad (4.2.5)$$
$$\dot{z} = - \beta z + xy.$$

This system is a Galerkin projection of a set of partial differential equations for two- dimensional. We shall study the bifurcation of (4.2.5) occurring at (x,y,z) = o and p = 1. the jacobian derivative at o is the matrix.

$$\begin{pmatrix} -\sigma & \sigma & O \\ p & -1 & O \\ O & O & -\beta \end{pmatrix} \qquad (4.2.6)$$

When $p = 1$, this matrix has eigenvalues 0, $-\sigma -1$ and $-\beta$ with eigenvectors (1,1,0) , (σ,- 1,0), (0,0,1). Using the eigenvectors as a basis for a new coordinate system, we set.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ w \end{pmatrix}, \quad \begin{pmatrix} U \\ V \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\sigma} & \frac{\sigma}{1+\sigma} & 0 \\ \frac{1}{1+\sigma} & \frac{-1}{1+\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Under this transformation (4.2.5) becomes

$$\dot{u} = \frac{1}{1+\sigma} \dot{x} + \frac{\sigma}{1+\sigma} \dot{y} = \frac{\sigma}{1+\sigma} (y - x) + \frac{\sigma}{1+\sigma} [(x - y) - xz]$$

$$= \frac{-\sigma}{1+\sigma} (u + \sigma v) w,$$

$$\dot{v} = \frac{1}{1+\sigma} \dot{x} - \frac{1}{1+\sigma} \dot{y} = \frac{\sigma}{1+\sigma} (y-x) - \frac{1}{1+\sigma} [x-y) - [x-y) - xz]$$

$$= - (1 + \sigma) v + \frac{1}{\quad} (u + \sigma v) w,$$

$$1 + \sigma$$

$$\dot{w} = \dot{z} = -\beta z + xy = -\beta w + (u + \sigma v)(u - v), \qquad (4.2.8)$$

or

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(1+\sigma) & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} U \\ V \\ w \end{pmatrix} + \begin{pmatrix} \frac{-\sigma}{1+\sigma}(u+\sigma v)w \\ \frac{-\sigma}{1+\sigma}(u+\sigma v)w \\ (u+\sigma v)(u-v), \end{pmatrix} \qquad (4.2.9)$$

so that the linear part is now standard (diagonal) form . In the (u, v,w) coordinates, the center manifold is a curve tangent to the u-axis. Note that the projection of the system onto the u-axis, obtained by setting v = w = 0 in the equation for u , yields u = 0 the u axis is not invariant, however , because the equation for w includes the term u^2. If we make a further monlinear coordinate change by setting $\tilde{w} = w - u^2/\beta$, however, we obtain.

$$\dot{\tilde{w}} = \dot{w} - \frac{2u\dot{u}}{\beta} = -\beta \left(w - \frac{u^2}{\beta} \right) + (\sigma-1)\, uv - \sigma v^2 + \frac{2\sigma}{\beta(1+\sigma)} u(u+\sigma v)w,$$

or

$$\dot{\tilde{w}} = -\beta\tilde{w} + (\sigma - 1)\, uv - \sigma v^2 + 2\, \frac{2\sigma}{\beta(1+\sigma)} u(u+\sigma v) \left(\tilde{w} + \frac{1}{r} \right). \quad \textbf{(4.2.10)}$$

In the $(u, v, \tilde{w})$ coordinate system. We have

$$\dot{u} = -\frac{\sigma}{1+\sigma} (u + \sigma v) \left(\tilde{w} + \frac{u^2}{\beta} \right) \qquad \textbf{(4.2.11)}$$

Now projection of the equation onto the u-axis in these coordinates gives the equation $\dot{u} = (-\sigma / \beta(1+\sigma))\, u^3$. Note also that

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no terms of the form u^2 occur in the equations for v and w , and thus that the u-axis is invariant in our transformed equations" up to second order".

Further efforts to find the center manifold can proceed by additional coordinate changes that serve to make the u-axis invariant for the flow.

This can be done iteratively by changes in v and $\tilde{w}$ which add to these coordinates mononomials in u, just as $\tilde{w}$ was obtained from w. Additional such coordinate changes will not change the coefficient (- σ /β(1 + σ) of u3 in the equation for u, but will affect higher degree terms of the from um, m ≥ 4. We shall see in subsequent sections that the equation u=(- σ /β(1 + σ) u3 along with the effect of varying p near 1 is sufficient to deduce the qualitative dynamics of the bifurcation in the Lorenz system (and the fluid system we started with). It is clearly important to include the calculation of the initial portion of the Taylor series of the center manifold in this analysis. Failure to do so gives a misleading picture of the dynamics at the point of bifurcation.

In studying the Lorenz example we have really been trying to approximate the (one- dimensional) equation governing the flow in the center manifold. We shall now develop a systematic method for performing such approximations.

The center manifold theorem implies that the bifurcating system is locally topologically equivalent to

$$\dot{\tilde{x}} = \mathcal{f}(x)$$

$$\dot{\tilde{y}} = -\tilde{y}\ ; \qquad (\tilde{x}, \tilde{y}, \tilde{z}) \in W^c \, x W^s \, x W^u ,$$

$$\dot{\tilde{z}} = \tilde{z}$$

at the bifurcation point. We now tackle the problem of computing the " reduced" vector field $\tilde{f}$. For simplicity, and because it is the most interesting case physically, we assume that the unstable manifold is empty and that the linear part of the bifurcating system is in block diagonal form:

$$\dot{x} = Bx + f(x, y),$$
$$\dot{y} = Cy + g(x, y) \quad , \qquad (x, y) \in R^n x R^m, \tag{4.2.13}$$

where B and C are n x n and m × m matrices whose eigenvalues have, respectively , zero real parts and negative real parts, and f and g vanish, along with their first partial derivatives, at the origin,

Since the center manifold is tangent to E^c (the y = 0 space) we can represent it as s (local) graph

$$W^c = \{ (x,y) / y= h(x)\}; \qquad h(0) = Dh(0) = 0, \quad (4.2..14)$$

Where $h: U \longrightarrow R^m$ is defined on some neighborhood $U \subseteq R^n$ of the origin. Figure (5). We now consider the projection of the vector field on y = h (x) onto E^c:

$$\dot{x} = Bx + f(x, h(x)). \qquad (4.2.15)$$

Since h (x) is tangent to y = O, the solutions of equation (4.2.15) provide a good approximation of the $\dot{\tilde{x}} = \tilde{f}(\tilde{x})$ restricted to W^c. In fact we have.

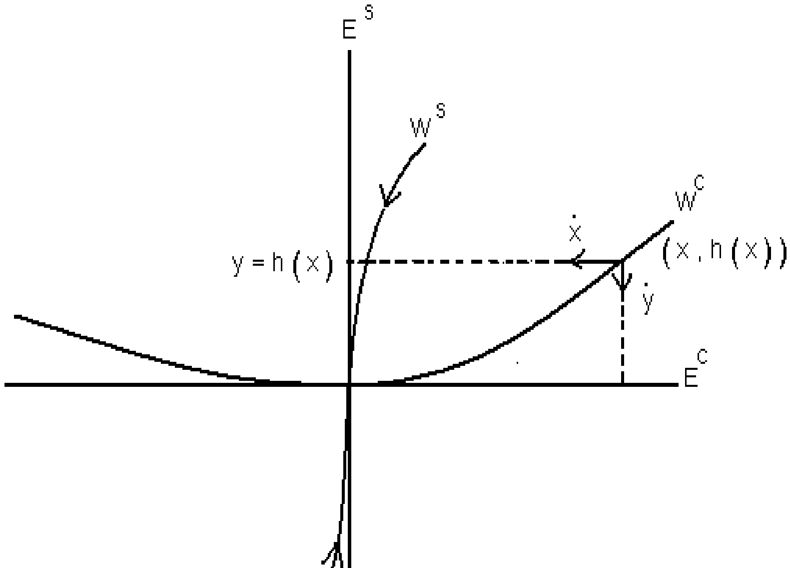

Figure (5) The center manifold and the projected vector field

Theorem 4.2.2 (Henry [1981], Carr [1981]. If the origin $x = 0$ of (4.2.15) is locally asymptotically stable (resp. unstable) then the origin of (4.2.13) is also locally asymptotically stable (resp. unstable).

This result also follows from the global linearization theory of Pugh and Shub [1970].

We now show how h(x) can be calculated, or at least approximated. Substituting y = h (x) in the second component of (4.2.13) and using the chain rule, we obtain.

$$\dot{y} = Dh(x)\,\dot{x} = Dh(x)[Bx + f(x,h(x))] = Ch(x) + g(x,h(x)),$$

Or

$$N(h(x)) = Dh(x)[Bx + f(x,h(x))] = Ch(x) - g(x,h(x)) = 0,$$

with boundary conditions

$$h(0) = Dh(0) = 0.$$

This (partial) differential equation for *h* cannot, of course, be solved exactly in most cases (to do so would imply that a solution of the original equation had been found), but its solution *can* be approximated arbitrarily closely as a Taylor series at $x = 0$:

Theorem 4.2.3 (Henry [1981], Carr [1981]. If a function Ø (x), with Ø(0) = DØ(0) = 0, can be found such that N(Ø(x)) = 0($|x|^p$) for some p > 1 as |x| → 0 then it follows that

$$h(x) = Ø(x) + O(|x|^p) \quad \text{as } |x| \to 0.$$

4.3 Normal Forms

In this section we continue the development of technical tools which provide the basis for our study of the qualitative properties of flows near a bifurcation. We assume that the center manifold theorem has been applied to a system and henceforth we restrict our attention to the flow within the center manifold. That is, to the approximating equation (4.2.15). we shall try to find additional coordinate transformations which simplify the analytic expiration of the vector field on the center manifold. The resulting "simplified" vector fields are called normal forms. Analysis of the dynamics of the normal forms yields a qualitative picture of the flows of each bifurcation type.

We now describe in more detail the problem of calculating the normal forms. We start with a system of differential equations.

$$\dot{x} = f(x), \qquad (4.3.1)$$

which has an equilibrium at 0. (In (4.3.1) we omit explicit reference to the parameter μ). We would like to find a coordinate change x = h (y) with h(0) =0 such that the system (4.3.1) becomes "as simple as possible." In the y-coordinates, we have

$$Dh(y)\dot{y} = f(h(y))$$

Or

$$\dot{y} = (Dh(y))^{-1} f(h(y)). \qquad (4.3.2)$$

The best that one can hope for is that (4.3.2) will be linear. Formally (this means in terms of power series), one can try to iteratively find a sequence of coordinate transformations h_1, h_2, … which remove terms of increasing degree from the Taylor series of (4.3.2) at the origin. The normal form procedures systematize these calculations without, however, giving the strongest results in all cases. In general, "as simple as possible" means that all inessential terms have been removed (up to some degree) from the Taylor series. When the procedure is applied to a hyperbolic equilibrium, then one obtains the formal part of Hartman's linearization theorem, as we now explain. After this digression, we shall return to nonhyperbolic bifurcating equilibria.

Assume for the moment that $Df(0)$ has distinct(but possibly complex) eigenvalues $\lambda_1\ldots\ldots,\lambda_n$ and that initial linear change of coordinates has diagonalized $Df(0)$. Then (4.3.1) written in coordinates becomes

$$\begin{aligned}\dot{x}_1 &= \lambda_1 x_1 + g_1(x_1, ..., x_n)\\ \dot{x}_2 &= \lambda_2 x_2 + g_2(x_1, ..., x_n) \qquad \text{or} \qquad \dot{x} = Ax + g(x), \qquad (4.3.3)\\ &\vdots\\ \dot{x}_n &= \lambda_n x_n + g_n(x_1, ..., x_n)\end{aligned}$$

where the functions g_i vanish to second order at the origin. We would like to find a coordinate change h of the form identity plus higher order terms, which has the property that (4.3.2) has non- linear terms which vanish to higher order than those of g. If k is the smallest degree of a nonvanishing derivative of some g_i we try to find a transformation h of the form

$$x = h(y) = y + P(y), \qquad (4.3.4)$$

with P a polynomial of degree k, so that the lowest degree of the nonlinear terms in the transformed equation (4.3.3) is $(k + 1)$. Now (4.3.3) takes the form

$$\dot{y} = (1 + DP(y))^{-1} f(y + p(y)). \qquad (4.3.5)$$

We want to expand this expression, retaining only terms of degree k and lower. Denoting the terms of g_i of degree k by g^k_i and P (y) by $(P_1(y), \ldots, P_n(y))$, we have

$$\dot{y}_i = \lambda_i y_i + \lambda_i P_i(y) + g^k_i(y) - \sum_{j=1}^{n} \frac{\partial P_i}{\partial y_j} \lambda_j y_j \qquad (4.3.6)$$

We have used in this formula the fact that $(1 + DP)^{-1} = 1 - DP$, modulo terms of degree k and higher. (To compute (4.3.5) modulo terms of degree (k+1) we only need $(1+DP)^{-1}$ modulo terms of degree k because f has degree1.). Therefore, we want to find a P which satisfies the equation.

$$\lambda_i P_i(y) - \sum_j \frac{\partial P_i}{\partial y_i} \lambda_i y_i = -g_i^k(y). \qquad (4.3.7)$$

we observe that the operator which associates to P the left-hand side of (4.3.7) is linear in the coefficients of P. In addition if P_i is the monomial $y_1^{a_1} \ldots y_n^{a_n}$, then $(\partial P_i/\partial y_j)\lambda_j y_j = a_j \lambda_j P_i$ and the left-hand side of (4.3.7) becomes $(\lambda - \sum_j a_j \lambda_j)P_i$ and hence the monomials are eigenvectors for the operator with eigenvalues $\lambda_i - \sum_j a_j \lambda_j$. We conclude that ***P*** can be found satisfying (4.3.7) provided that none of the sums $\lambda_i - \sum_j a_j \lambda_j$ is zero when $a_1, \ldots, a_n$ are nonnegative integers with $\sum_j a_j$ = k. If there is no equation $\lambda_i - \sum_j a_j \lambda_j = 0$ which is satisfied for nonnegative integers a_j with $\sum_j a_j \geq 2$, then the equation can be linearized to any desired algebraic order[5].

For bifurcation theory we are specifically interested in equilibria at which there are eigenvalues with zero real parts. At such equilibria, the linearization cannot be solved and there are (nonlinear)resonance terms in f which cannot be removed by coordinate changes. The normal form theorem formulates systematically how well one can do, using the procedure analogous to that outlined above to solve the linearization problem for hyperbolic equilibria. The key observations which form the basis of the computations are : (1) that the solvability depended only on the linear part of the vector field; and(2) that the problem can be reduced to a sequence of linear equation to be solved. The final result is a Taylor series for the vector field which contains only the essential resonant terms.

If L=Df(0) x denotes the linear part of (4.3.1) at x = 0, then L duces a map ad L on the linear space H_k of vector fields whose coefficients are homogeneous polynomials of degree k. The map ad L is defined by

$$\text{ad } L(Y) = [Y, L] = DLY - DYL, \qquad (4.3.8)$$

where [.,.]denotes the Lie bracket operation (Abraham of and Marsden [1978], Choquet-Bruhat et al [1977]). In component form, we have

$$[Y,L]^i = \sum_{j=1}^{n} \left(\frac{\partial L^i}{\partial y_i} Y^j - \frac{\partial L^i}{\partial y^i} L^j \right) . \qquad (4.3.9)$$

The main result is

Theorem4.3.1.Let $\dot{x} = f(x)$ be a C^r system of differential equations with $f(0) = 0$ and $Df(0)x = L$. choose a complement G_k for ad L (H_k) in H_k, so that H_k = ad L $(H_k) + G_k$. then there is an analytic change of coordinates in a neighborhood of the origin which transforms the system $\dot{x} = f(x)$ to $\dot{y} = g(y) = g^{(1)}(y) + g^{(2)}(y) + \ldots\ldots.+ g^{(r)}(y) + R_r$ with $L = g^{(1)}(y)$ and $g^{(k)} \varepsilon G_k$ for $2 \leq k \leq r$ and $R_r = 0(|y|^r)$.

Proof. We give a constructive proof which can be used to implement the calculations of normal forms in examples. The procedure follows the pattern in our discussion of the linearization problem. We use induction and assume that $\dot{x} = f(x)$ has been transformed so that the terms of degree smaller than s lie in the complementary subspace G_i , $2 \leq i < s$. we then introduce a coordinate transformation of the form $x = h(y) = y + P(y)$, where P is a homogeneous polynomial of degree s whose coefficients are to be determined. Substitution then gives the equation.

$$(I + DP(y))\dot{y} = f^{(1)}(y) + f^{(2)}(y) + \cdots + f^{(s)}(y) + Df(0)P(y) + 0(|y|^s). \quad (4.3.10)$$

The terms of degree smaller than s are unchanged by this transformation, while the new terms of degree s are

$$f^{(s)}(y) + DLP(y)L - DP(y)L = f^{(s)}(y) + \text{ad } L(P)(y), \quad (4.311)$$

where $L(y) = f^{(1)}(y)$. Clearly a suitable choice of P will make

$$f^s(y) + \text{ad } L(P)(y)$$

lie in G_s as desired [5].

4.4 Codimension One Bifurcations Of Equilibria

In this section we describe the simplest bifurcations equailibria. These are represented by the following four differential equations which depend on a single parameter μ:

$$\dot{x} = \mu - x^2 \qquad \text{(saddle-node)}, \quad (4.4.1)$$

$$\dot{x} = \mu x - x^2 \qquad \text{(transcritical)}, \quad (4.4.2)$$

$$\dot{x} = \mu x - x^3 \qquad \text{(pitchfork)}, \quad (4.4.3)$$

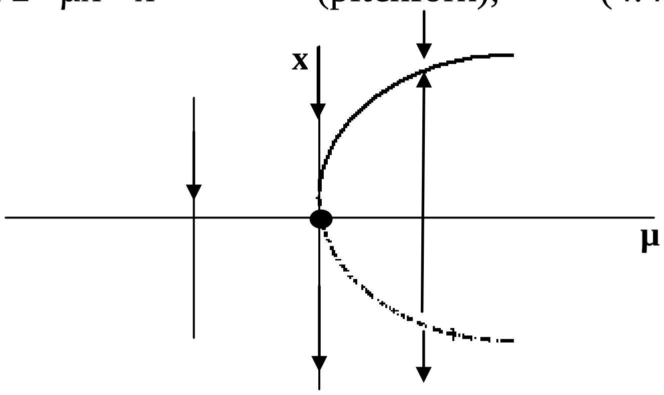

Figure (6) saddle-node bifurcation.

and

$$\begin{cases} \dot{x} = -y + x(\mu - (x^2 + y^2)) \\ \dot{y} = x + y(\mu - (x^2 + y^2)) \end{cases} \qquad \textit{(Hopf)} \quad (4.4.4)$$

The bifurcation diagrams for these four equations are depicted in Figures (6)- (9). Each of the equations (4.4.1) – (4.4.4) arises naturally in a suitable context as determining the local qualitative behavior of the generic bifurcation of an equilibrium. Our purpose here is to describe in detail how, and under what conditions, one can reduce the study of the general equation (4.1.1) to one of these four specific examples.

The Saddle-Node

Consider a system of equations.

$$\dot{x} = f_{\mu}(x), \qquad (4.4.5)$$

With x ε R^n,μ ε R, and f_μ smooth. Assume that at $\mu = \mu_0$, $x = x_0$, (4.4.5) has an equilibrium at which there is a zero eigenvalue (for the linearization). Usually, this zero eigenvalue will be simple, and the center manifold theorem allows us to reduce the study of this kind of bifurcation problem to one in which x is one dimensional.

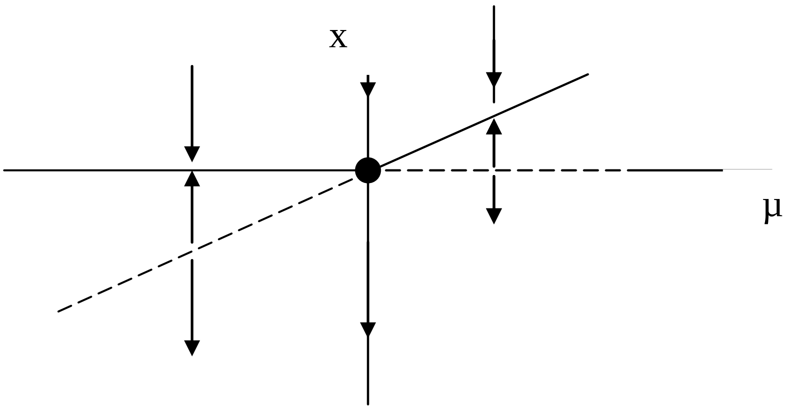

Figure (7). Transactional bifurcation.

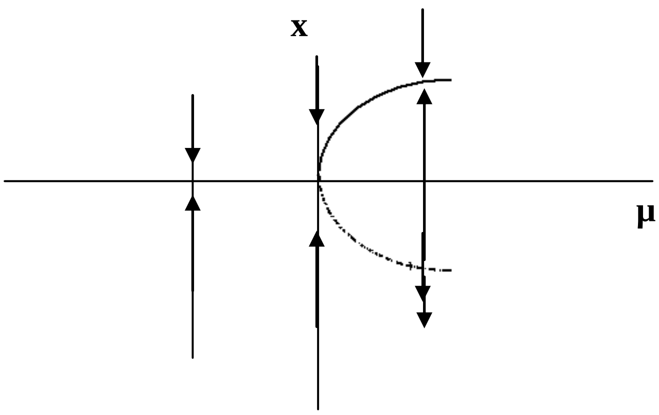

Figure (8)Pitchfork bifurcation(supercritical).

we can find a two- dimensional center manifold $\sum \sqsubset R^n \times R$ passing through (x_0,μ_0) such that.

1- The tangent space of $\sum$ at $(x_0,\mu_0)$is spanned by an eigenvector of 0 for $Df_{\mu o}$ (x_0) and a vector parallel to the μ- axis.
2- For any finite r,$\sum$ is C^r if restricted to a small enough neighborhood of(x_0,μ_0).
3- The vector field of (4.4.5) is tangent to$\sum$.
4- There is a neighborhood U of (x_0,μ_0)in $\mathbf{R}^n \times \mathbf{R}$ such that all trajectories contained entirely in U for all time lie in$\sum$.

(Note: The center manifold theorem allows one to formulate stronger properties than (4) which describe the qualitative structure of trajectories which remain close to (x_0,μ_0) in forward time or in backwards time , cf. Carr [1981].

Restricting (4.4.5) to Σ we obtain a one-parameter family of equations on the one-dimensional curves $\sum_\mu$ in $\sum$ obtained by fixing μ[5]This one-parameter family is our reduction of the bifurcation problem.

Let us now formulate transversality conditions for a system (4.4.5) with n = 1 , which yield the saddle-node bifurcation. We have $(df_{\mu 0} / dx)(x_0) = 0$, but we take $(\partial f_{\mu 0}/\partial\mu)(x_0) \neq 0$ as a transversaliy condition. The implicit function theorem then implies that the equilibrai of (4.4.5) form a curve which will be tangent to the line $\mu=\mu_0$. An additional transversality condition $(d^2 f_{\mu o}/dx^2)(x_o)\neq 0$ implies that the curve of equilibria has a *quadratic* tangency with $\mu=\mu_0$ and locally lies to one side of this line.

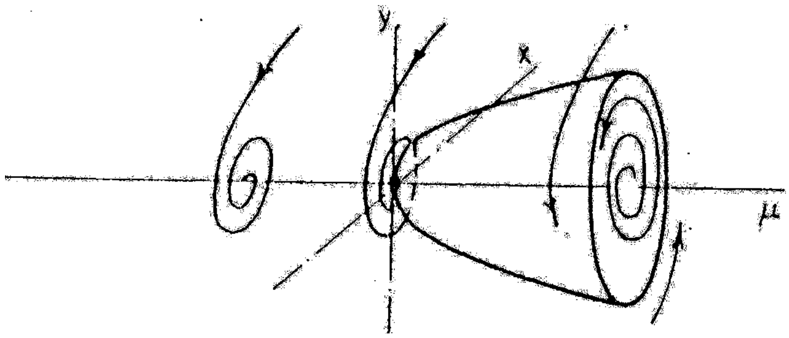

Figure (9). Hopf bifurcation (Supercritical).

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phase portraits of this system are topologically equivalent to those of a family $\dot{x} = \pm(\mu-\mu_0)\pm(x-x_0)^2$.However, we can also formulate these transversality conditions for an n-dimensional system without recourse to center manifold reduction. The following theorem states the necessary conditions [5].

Theorem4.4.1. Let $x = f_\mu(x)$be a system of differential equations in $\mathbf{R}^n$ depending on the single parameter μ. When $\mu=\mu_0$, assume that there is an equilibrium *p* for which the following hypotheses are satisfied:

(SNI) has $D_x f_{\mu o}(p)$ has a simple eigenvalue 0 with right eigenvector v and left w.$D_x f_{\mu 0}$ (p) has k eigenvalues with negative real parts and (n –k – 1) eigenvalues with positive ral parts (coumting multiplicity)

(SN2) $w((\partial f_\mu / \partial\mu)(p,\mu_0))\neq 0$.

(SN3) $\ddot{w}(D_x^2 f_{\mu o}(P)\upsilon,\upsilon)\neq 0$.

Then there is *a* smooth curve of equilibria $\mathbf{R}^n \times \mathbf{R}$ passing through (p,μ_0), tangent to the hyperplane $R^n \times \{\mu_0\}$.Depending on the signs of expressions in (SN2) and (SN3) there are no equilibria near $(p,\mu_0)$when $\mu<\mu_0$ ($\mu>\mu_0$) and two equilibria near (p,μ_0) for each parameter value $\mu>\mu_0$ ($\mu<\mu_0$). The two equilibria for $x = f_\mu(x)$ near

(p,μ_0) are hyperbolic and have stable manifolds of dimensions k and k + 1, respectively. The set of equations $x = f_\mu(x)$ which satisfy (SN1) – (SN3) is open and dense in the space of C^∞ one- parameter families of vector fields with an equilibrium at (p,μ_0) with a zero eigenvalue.

This formal (and formidable) therorem merely expresses the fact that the " generic" saddle node bifurcation is qualitatively like the family of equations $\dot{x} = \mu - x^2$ in the direction of the zero eigenvector, with hyperbolic behavior in the complementary directions. Hypotheses (SN2) and (SN3) are the transversality conditions which control the non degeneracy of the behavior with respect to the parameter and the dominant effect of the quadratic nonlinear term [5].

The result obtained from theorem 4.4.1 are limited in two different ways. On the one hand, it is possible that more quantitative information about the flows near bifurcation can be extracted. For example, one can use the system $\dot{x} = \mu - x^2$ to give estimates of how rapid the convergence to the various equilibria are. Higher- order terms in the Taylor expansion of an equation can be used to refine these estimates. This is an aspect of the theory of differential equations which we do not pursue further in this chapter because our attention is to focus on geometric issues rather than analytic ones. In this regard, we should be reminded that we often do not strive to state the strongest or most general theorem for a given situation but rather aim to illustrate the phenomena and methods of analysis in the simplest ways.

The second limitation of Theorem 4.4.1 is that there may be global changes in a phase portrait associated with a saddle- node bifurcation. Consider, for example, the flows depicted in Figure (10), which we have already met in the Van der Pol example[5].Here a saddle- node in a two- dimensional system occurs, with the coalescense of a sink and a saddle. After the bifurcation , there is a new periodic orbit which has appeared because the unstable separatrix at the saddle- node lies in the two-dimensional stable manifold of the bifurcating equilibrium. This is an example of a global bifurcation phenomenon that cannot be reduced to the study of a neighborhood of an equilibrium or a fixed point in a return map [5].

Transcritical and Pitchfork Bifurcations:

The importance of the saddle- node bifurcation is that all bifurcations of one- parameter families at an equilibrium with a zero eigenvalue can be perturbed to saddle-node bifurcations. Thus one expects that the zero eigenvalue bifurcations encountered in applications will be saddle- nodes.If they are not, then there is probably something special about the formulation of the problem which restricts the context so as to prevent the saddle-node from occurring. The transcritical bifurcation is one example which illustrates how the setting of the problem can rule out the saddle-node bifurcation.

In classical bifurcation theory, it is often assumed that there is a trivial solution from which bifurcation is to occur. Thus, the systems (4.4.5) are assumed to satisfy $f_\mu(0) = 0$ for all μ, so that x = 0 is an equilibrium for all parameter values. Since the saddle-node

Figure (10) A saddle node occurring on a closed curve leads to global bifurcation.

near the point of bifurcation, this situation is qualitatively different. To formulate the appropriate transversality conditions we look at the one- parameter families which satisfy the constraint that $f_\mu(0) = 0$ for all μ. This prevents hypothesis (SN2) of Theorem 4.4.1 from being satisfied. If we replace this condition by the requirement that $w((\partial^2 f \partial\mu\partial x)(\upsilon))\neq 0$ at $(0,\mu_0)$ then the phase portraits of the family near the bifurcation will be topologically equivalent to those of Figure(7) and we have a transcritical bifurcation or exchange of stability[5].

A second setting in which the saddle-onde does not occur involves systems which have a symmtry. Many physical problems are formulated so that equations defining the system do have symmetries of some kind. For example, the Duffing equation is invariant under the transformation (x ,y) → (- x, - y) and the Lorenz equiation is symmetric under the transformation (x ,y ,z) → (-x, -y, z).In one dimension, a differential equation(4.4.5) is symmetric or equivariant with respect to the symmetry x → −x if $f_\mu(-x) = -f_\mu(x)$. Thus the equivariant vector fields are ones for which f_μ is an odd function of x. In particular, all such equations have equilibrium at 0. The transcritical bifurcation cannot occur in these systems, however, because an odd function f_μ cannot satisfy the condition $\partial^2 f_\mu/\partial x^2 \neq 0$ required by the transcritical bifurcation (cf.SN3).if this condition is replaced by the transversality hypothesis $\partial^3 f_\mu/\partial x^3 \neq 0$, then one obtains the pitchfork bifurcation. At the point of bifurcation, the stability of the trivial equilibrium changes, and a new pair of equilibria (related by the symmetry) appear to one side of the point of bifurcation in parameter space, as in Figure(8) we leave to the reader the formulation of results analogous to The theorem 4.4.1 for the transcritical and pitchfork bifuracation (cf.Sotomayor[1973]) .

We note that the direction of the bifurcation and the stability of the branches in these example is determined by the sign of$\partial^2 f_\mu/\partial x^2$ $\partial^3 f_\mu/\partial x^3$. In the last case, if $\partial^3 f_\mu/\partial x^3$ is negative, then the branches occur above the bifurcation value and we have a supercritical pitchfork bifurcation, wherease we have a subcritical bifurcation if it is positive.

Hopf Bifurcations:

Consider now a system(4.4.5) with a parameter value μ_0 and equilibrium $p(\mu_0)$ at which $Df_{\mu o}$ has a simple pair of pure imaginary eigenvalues $\pm$ ίω، $\omega > 0$, and no other eigenvaluses with zero real part. The implicit function theorem guarantees (since $Df_{\mu o}$ is invertible) that for each μ near μ_0 there will be an equilibrium $p(\mu)$ near $p(\mu_0)$ which varies smoothly with μ. Nonetheless, the dimensions of stable and unstable manifolds of $p(\mu)$ do change if the eigenvalues of $Df(p(\mu))$ cross he imaginary axis at μ_0. This qualitative change in the local flow near $p(\mu)$ must be marked by some other local changes in the phase portraits not involving fixed points.

A clue to what happens in the generic bifurcation problem involving an equilibrium with pure imaginary eigenvalues can be gained from examining linear systems in which there is a change of this type. For example, consider the system

$$\begin{aligned} \dot{x} &= \mu x - \omega y, \\ \dot{y} &= \omega x + \mu y, \end{aligned} \qquad (4.4.6)$$

whose solutions have the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\mu t} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad (4.4.7)$$

When $\mu<0$, , solutions spiral into the origin, and when $\mu > 0$, solutions spiral away from the origin. When $\mu = 0$, all solutions are periodic. Even in a one- parameter family of equations, it is highly special to find a parameter value at which there is a whole family of periodic orbits, but there is still a surface of periodic orbits which appears in the general problem [5].

The normal form theorem gives us the required information about how the generic problem differs from the system (4.4.6) by smooth changes of coordinates, the Taylor series of degree 3 for the general problem can be brought to the following form (cf. Equation (4.3.15))

$$\begin{aligned} \dot{x} &= (d\mu+\boldsymbol{a}(x^2+y^2))x-(\omega+c\mu+b(x^2+y^2))y, \\ \dot{y} &= (\omega+c\mu+b(x^2+y^2))x+(d\mu+\boldsymbol{a}(x^2+y^2))y, \end{aligned} \qquad (4.4.8)$$

which is expressed in polar coordinates as

$$\begin{aligned} \dot{r} &= (d\mu+ar^2)r, \\ \dot{\theta} &= (\omega+c\mu+br^2). \end{aligned} \qquad (4.4.9)$$

Since the r equation in (4.4.9) separates from θ, we see that there are periodic orbits of (4.4.8) which are circles r = const., obtained from the nonzero solutions of $\dot{r} = 0$ in (4.4.9) If $a \neq 0$ and $d \neq 0$ these solutions lie along the parabola $\mu = -ar^2/d$. This implies that the surface of periodic orbits has a quadratic tangency with its tangent plane $\mu = 0$ in $R^2 \times R$. the content of the Hopf bifurcation theorem is that the qualitative properties of (4.4.8) near the origin remain unchanged if higher- order terms are added to the system:

Theorem 4.4.2 (Hopf [1942]). Suppose that the system $\dot{x} = f_\mu(x)$, $x \varepsilon R^n, \mu \varepsilon R$ has an equilibrium (x_o, μ_o)at which the following properties are satisfied :

(H1) $D_x f_{\mu o}(x_o)$ has a simple pair of pure imaginary eigevalues and no other eigenvalues with zero real parts.

Then (H1) implies that there is a smooth curve of equilibria$(x(\mu),\mu)$ with $x(\mu_0) = x_0$. The eigenvalues $\lambda(\mu)$, $\overline{\lambda}(\mu)$ of $D_x f_{\mu o}(x(\mu))$ which are imaginary at $\mu = \mu_0$ vary smoothly with μ. If, moreover.

(H2) $$\frac{d}{d\mu}(\mathrm{Re}\,\lambda(\mu))|_{\mu=\mu o} = d \neq 0,$$

Then there is a unique three- dimensional center manifold passing through (x_0,μ_0) in $R^n \times R$ and a smooth system of coordinates preserving the planes μ = const) for which the Taylor expansion of degree 3 on the center manifold is given by (4.4.8). If $a \neq 0$, there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_o),\lambda(\mu_o)$ agreeing to

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second order with the paraboloid $\mu = -(a/d)(x^2+y^2)$. If $a < 0$, then these periodic solutions are stable limit cycles, while if $a > 0$, the periodic solutions are repelling.

This theorem can be proved by a direct application of the center manifold and normal form theorems given above (cf. Marsden and McCracken [1976]).

For large systems of equations, computation of the nomral form(4.4.8) and the cubic coefficient *a*, which determines the stability, can be a substantial undertaking .

In a two- dimensional system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \qquad (4.4.10)$$

With $f(0) = g(0) = 0$ and $Df(0) = Dg(0) = 0$, the normal form calculation, which we sketch in the appendix to this section, yields

$$a = \frac{1}{16}[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \frac{1}{16\omega}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - (f_{xx}g_{xx} + f_{yy}g_{yy}], \qquad (4.4.11)$$

where f_{xy} denotes $(\partial^2 f/\partial x\partial y)(0,0)$, etc. In applying this formula to systems of dimension greater than two, however, the reader should recall that the quadratic terms which play *a* role in the center manifold calculations can affect the value of *a*. One cannot find a by simply projecting the system of equations onto the eigenspace of $\pm i\omega$, but must approximate the center manifold at least to quadratic terms .

4.5.Codimension One Bifurcations of Maps and Periodic Orbits

In this section we consider the simplest bifurcations for periodic orbits. The strategy that we adopt involves computing Poincare' return maps and then trying to repeat the results of Section 4.4 for these discrete dynamical systems. There are some additional complications that introduce new subtleties to some of these problems. In practice, computations of the bifurcatins of periodic orbits from a defining system of equations are substantially more difficult than those for equilipria because one must first integrate the equations near the periodic orbit to find the poincare' return map before further analysis can proceed. Thuse, the result obtained here have been most frequently applied:

(1) in comparison with numerical calculations;

(2) directly to discrete dynamical systems defined by a mapping; or

(3) in perturbations situations close to ones in which a system can be explicitly integrated.

In view of these computational difficulties, in this section we shall focus upon the geometric aspects of these bifurcations.

There are three ways in which a fixed point *p* of a discrete mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ may be fail to be hyperbolic: D *f* (p) may have an eigenvalue +1, an eigenvalue – 1, or a pair of complex eigenvalue $\bar{\lambda}, \lambda$ with $|\lambda| = 1$.(If *Df (p)* has an eigenvalue μ at the fixed point *p*, we say *p* has eigenvalue μ.)

The bifurcation theory for fixed points with eigenvalue 1 is completely analogous to the bifurcation theory for equilibria with eigenvalue 0. The generic one-parameter family has a tow-dimensional center manifold (including the parameter direction) on which it is topologically aquivalent to the saddle-node family defined by the map.

$$f_\mu (x) = x + \mu - x^2 \qquad (4.5.1)$$

The same considerations of constraint and symmetry as discussed in previously alter the generic picture, giving either transcritical or pitchfork bifurcations.

Bifurcations with eigenvalue – 1 do not have an analogue for equilibria, while the theory for complex eigenvalues is more subtle than that of the Hopf bifurcation for flows.

Eigenvalues with – 1 are associated with *flip* bifurcations, also referred to as *period doubling* or *subharmonic* bifurcations. Using a center manifold reduction, we restrict our attention to one-dimensional mappings f_μ and assume that μ is a one-dimensional parameter. If 0 is a fixed point of $f_{\mu o} : R \rightarrow R$ with eigenvalue -1, then the Taylor expansion of $f_{\mu o}$ to degree 3 is

$$f_{\mu o} (x) = -x + a_2 x^2 + a_3 x^3 + R_3 (x), \text{ with } R_3(x)=0 (|x^3|).(4.5.2)$$

The implicit function theorem guarantees that there will be a smooth curve (x(μ),μ) of fixed points in the plane passing through $(0,\mu_o)$, so, apart from a change of stability, we must look for changes in the dynamical behavior elsewhere. Composing $\boldsymbol{f}_{\mu o}$ with itself, we find

$$f^2_{\mu o}(x) = -(-x + a_2 x^2 + a_3 x^3) + a_2 (- x + a_2 x^2)^2 + a_3 (-x)^3 + R_3$$

$$= x - (2a_2^2 + 2a_3) x^3 + \tilde{\tilde{R}}_3. \qquad (3.5.3)$$

Since $f^{2}_{\mu o}$ has eigenvalue + 1, its fixed points need not vary smoothly and we expect that there may be fixed points of f^{2}_{μ} near $(0,\mu_o)$ which are not fixed points of f_μ. Such points are evidently periodic orbits of period 2.

Examining the Taylor series of $f^{2}_{\mu o}(x)$, we see that the coefficient of the quadratic term is zero, and thus the bifurcation behavior has similarities with the pitchfork, the primary difference being that the new orbits which appear are not fixed points but have period 2. The ideas outlined above lead to the following result.

Theorem4.5.1. Let $f_\mu : \mathbb{R} \rightarrow \mathbb{R}$ be a one-parameter family of mappings such that $f_{\mu o}$ has a fixed point x_o with eigenvalue –1. Assume

$$\text{(F1)} \quad \left(\frac{\partial f}{\partial \mu} \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial \mu} \right) = \frac{\partial f}{\partial \mu} \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial f}{\partial x} - 1 \right) \frac{\partial^2 f}{\partial^2 X \partial \mu} \neq 0 \text{ at } (x_o,\mu_o);$$

$$\text{(F2)} \qquad a = \left[\tfrac{1}{2} \left(\frac{\partial^2 f}{\partial x^2} \right)^2 + \tfrac{1}{3} \left(\frac{\partial^3 f}{\partial x^3} \right) \right] \neq 0 \text{ at } (x_0,\mu_0).$$

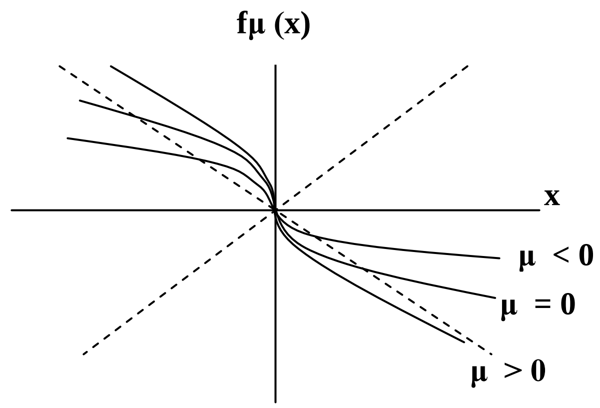
Then there is a smooth curve of fixed point of f_μ passing through (x_o,μ_o) the stability of which change at (x_0,μ_0). There is also *a* mooth curve γ passing through (x_o,μ_o) so that $\gamma - \{(x_o,\mu_o)\}$is a union of hyperbolic period 2 orbits. The curve γ has quadratic tangency with the line $\mathbb{R} \times \{\mu_o\}$at (x_o,μ_o).

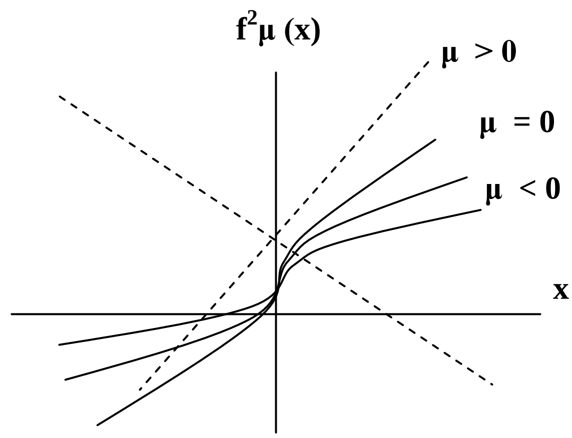
Here the quantity (F1) is the μ-derivative of *f* along the curve of the fixed points. It plays the role of the nondegeneracy conditions SN2 and H2 in Theorems 4.4.1 and 4.4.2. In (F2) the sign of *a* determines the stability and direction of bifurcation of the orbit of period 2. If *a* is positive, the orbits are stable; if *a* is negative they are unstable. We note that cubic terms $(\partial^3 f/\partial x^3)$ are necessary for the determination of *a*.

Figure (11) shows the bifurcation diagram for the family.

$$f_\mu(x) = -(1+\mu)x + x^3. \qquad (4.5.4)$$

(a)





(b)

(c)

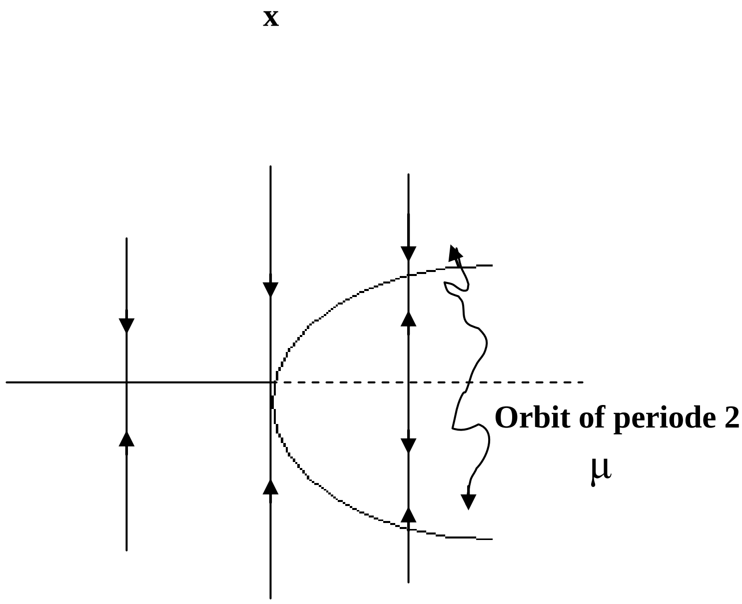


Figure (11). The flip bifurcation for equation (4.5.4). (a) Graphs of f_μ (x); (b) graphs of f_μ^2(x); the bifurcation digram.

As an example we consider the one-dimensional quadratic map

$$f_\mu : x \longrightarrow \mu - x^2 \qquad (4.5.5)$$

The upper branch of equilibria is given (for $\mu > -\frac{1}{4}$) by

$$x = -\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mu} \quad . \qquad (4.5.6).$$

Linearizing along that branch, we find

$$\frac{\partial f}{\partial x} = -2x = 1 - \sqrt{1 + 4\mu} \qquad (4.5.7)$$

and evidently $\partial f / \partial x = -1$ at $\mu = \frac{3}{4}$, and hence $(x_0, \mu_0) = (\frac{1}{2}, \frac{3}{4})$ is a candidate for a flip bifurcation point. In this example it is easy to check that conditions F1 and F2 of Theorem 4.5.1 are met and hence that a flip occurs.

The stability of the period two orbits in this example is determined by noting that the second and third derivatives of *f* at (x_{0},μ_0) are, respectively

$$\frac{\partial^2 f}{\partial \mathbf{x}^2}(x_0,\mu_0) = -2 \quad \text{and} \quad \frac{\partial^3 f}{\partial \mathbf{x}^3} \equiv 0, \qquad (4.5.8)$$

and hence that the quantity *a* of (F2) is positive: the flip is supercritical.

We make one final remark about the relationship of return map *P* with eigen value -1 at a fixed point *p*, to continuous flow around the corresponding periodic orbit. The trajectories of *p* alternate from one side of *p* to the other along the direction of the eigenvector to – 1 (cf. section 3.4, table 3.4.1). This means that the tow-dimensional center manifold for the periodic orbit is twisted around the periodic orbit like Mobius band around its center line. The map *P* which glues the two ends of a strip together reverses orientation around *p*.. One cannot embed a Mobius strip in an orientable tow-dimensional manifold (such as the plane), so that the flip bifurcation cannot occur in such systems 5 . As we shall see, however, flip bifurcations can and do occur in flows of dimension ≥ 3.

We now turn to bifurcations of a periodic orbit at which there are complex eigenvalues $\lambda,\bar{\lambda}$ with $|\lambda| = 1$. Analogy with the theory of Hop bifurcation suggests that orbits near the bifurcation will be present which encircle the fixed point. An individual orbit of a discrete mapping cannot fill an entire circle and the bifurcation structure is more complicated than that which can be deduced from a search for new periodic orbits. Indeed, there are flows near the bifurcation which have no new periodic orbits near the bifurcating one but have quasiperiodic orbits instead. A more subtle analysis is required to capture these. Before one reaches this portion of the analysis, however, there is another difficulty to contend with.

Let us assume that we have a transformation $f : \mathbf{R}^2 \to \mathbf{R}^2$ so that the origin is a fixed point and $Df(0)$ is the matrix which is rotation by the angle $2\pi\theta$:

$$\begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ ++\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} \qquad (4.5.9)$$

We want to perform normal form calculations which simplify the higher-order terms in the Taylor series of f by using nonlinear coordinate transformation. As in the case of flows, the calculations are simplified if they are complexifed (cf. the Appendix to section 4.4). If we regard (x,y) as each being complex, then the egenvectors of Df (0)are

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{with}$$

eigenvalues $e^{2\pi i\Theta}$ and $e^{-2\pi i\Theta}$ and coordinates z and $\overline{z}$, respectively Suppose we now want to alter the Taylor expansion at degee *k* by a real coordinate transformation of the form.

$$h\,(z,\overline{z}) = \text{id} + \text{terms of degree } k.$$

Since the z coordinate of the image of *h* is the complex conjugate of the *z* coordinate of the image of *h*, it suffices to compute the *z* coordinate of the image of the conjugated

$$\text{ad } Df \begin{pmatrix} z^1 \;\; z^{k-1} \\ \\ 0 \end{pmatrix} = (e^{2\pi i\,(21-k)\Theta} - e^{2\pi i\Theta}) \begin{pmatrix} z^1 \;\; z^{k-1} \\ \\ 0 \end{pmatrix}, \tag{4.5.14}$$

$$\text{ad } Df \begin{pmatrix} 0 \\ \\ z^1 \;\; z^{k-1} \end{pmatrix} = (e^{2\pi i\,(21-k)\Theta} - e^{-2\pi i\Theta}) \begin{pmatrix} 0 \\ \\ z^1 \;\; z^{k-1} \end{pmatrix},$$

The zero eigenvalues of Ad D*f* occur when $(2l - k)\,\Theta \equiv \pm\,\Theta$ (mod1). If Θ is irrational, then this can happen only when *k* is odd and $l = (k \pm 1)/2$. the zero eigenvectors for these values

have the form $(\overline{z}z)^1 \begin{pmatrix} z \\ \\ 0 \end{pmatrix}$ and $(\overline{z},z)^1 \begin{pmatrix} 0 \\ \\ z \end{pmatrix}$.

Expressed in real terms these represent mappings of the form $(x^2 + y^2)^1$

$$\begin{pmatrix} a & -b \\ \\ & \end{pmatrix}$$

g(x,y) where **g** is a linear mapping with matrix

b a When

Θ is irrational, therefore, the normal forms of f are analogues of the normal forms computed for the Hopf bifurcation for flows. However, if Θ is rational, then there are additional resonant terms witch come from other solutions of the equation (21 – k)Θ

≡ ± Θ(mod 1). The denominator of Θ determines the lowest degree at witch these terms may appear.

We have already met the cases Θ = 0 (saddle-node) and Θ= ½ (flip).

In addition, when Θ = ±⅓ or Θ = ±¼ , then there are resonant terms of degree 2, or 3, respectively in normal forms.

When Θ = 0 ±⅓ these have the complex form $\bar{z}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\bar{z}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

while if Θ = ± ¼ these have the form $\bar{z}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\bar{z}^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ This means that the bifurcation structures associated with fixed points that are third and fourth root of unity are special. Arnold [1977] and Takens [1974b] present analyses of these cases. If one assumes that λ is not a third or fourth root of unity, then it is possible to proceed with a general analyses of Hopf bifurcation for periodic orbits (secondary Hopf bifuceation), and we have:

theorem 4.5.2. Let $f_\mu : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be *a* one-parameter family of mapping which has *a* smooth family of fixed points x(μ) at which the eigenvalues are complex conjugates λ (μ), $\lambda(\mu)$. Assume

(SH1) $|\lambda(\mu_0)| = 1$ but $\lambda^j(\mu_0) \neq 1$ for $j = 1, 2, 3, 4$.

(SH2) $\frac{d}{d\mu}(|\lambda(\mu_0)|) = d \neq 0.$

Then there is *a* smooth change of coordinates *h* so that the expression of $hf_\mu h^{-1}$ in polar coordinate has the form

$$hf_\mu h^{-1}(r,\Theta)=(r(1+d(\mu-\mu_0)+ar^2),\Theta+c+br^2)+\text{higher-order terms.} \tag{4.5.15}$$

(Note:λ complex and (SH2) imply $|\arg(\lambda)| = c$ and d are nonzero.) If, in addition

(SH3) $a \neq 0$

Then there is *a* two-dimensional surface $\sum$ (not necessarily infinitely differ-entiable) in $R^2 \times R^2$ having quadratic tangency with the plane $R^2 \times \{\mu_0\}$.

Which is invariant for *f* .I *f* $\sum \cap (R^2 \times \{\mu\})$ is larger than apoint, then it is a simple closed curve.

As in the case of flows, the signs of the coefficients *a* and *d* determine the direction and stability of the bifurcating periodic orbits, *c* and *b* give asymptotic information on rotation numbers, as outlined below.

Marsden and McCracken [1976] contains Lanford's exposition of Ruelle's proof of this theorem using the technique of graph transforms. The theorem states that (outside the strong resonance cases $\lambda^3 = 1$ and $\lambda^4 = 1$), something like the limit cycles of the Hopf theorem appear in the phase portrait of f_μ. these are simple closed curves which bound the basin of attraction or repulsion of a fixed point. Theorem 4.5.2 dose not however address the question of describing the dynamics within ∑. In all of it's details, this is a difficult problem which involves the introduction of *rotation numbers* and consideration of subtle small divisor problem. 5 . Here we comment only that if $b \neq 0$ *in* (4.5.15) then it can be proved that there will be a complicated pattern of periodic and quasiperidic dynamical behavior on ∑.To study this one must examine the global bifurcations of diffeomorphisms whose state space is the circle.

A stability formula, giving an expression for the coefficient *a* in the normal form (4.5.15), can be derived in much the same way as for flows. For details see looss and Joseph [1981] or Wan [1978]. Assuming that the bifurcating system (restricted to the center manifold) is in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \cos(c) & -\sin(c) \\ \sin(c) & \cos(c) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \qquad \textbf{(4.5.16)}$$

with eigenvalues $\lambda, \bar{\lambda} = e^{\pm ic}$, on obtains

$$\begin{bmatrix} (1-2\lambda)\bar{\lambda}^2 \\ \\ \end{bmatrix}$$

$$a = -\mathrm{Re}\ \frac{\qquad}{1-\lambda}\ \xi_{11}\xi_{20}\ -½\ |\xi_{11}|^2 - |\xi_{02}|^2 + \mathrm{Re}\ (\bar{\lambda}\xi_{21}),$$

where

$$\xi_{20} = ⅛[(f_{xx} - f_{yy} + 2g_{xy}) + i\ (g_{xx} - g_{yy} - 2f_{xy})],$$

$$\xi_{11} = ¼[(f_{xx} + f_{yy}) + i\ (g_{xx} + g_{yy})],$$

$$\xi_{02} = ⅛[(f_{xx} - f_{yy} - 2g_{xy}) + i(g_{xx} - g_{yy} + 2f_{xy})],$$

and

$$\xi_{21} = \frac{1}{16}\ [(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + i(g_{xxx} + g_{xyy} - f_{xxy} - f_{yyy})]. \qquad (5.4.17)$$

We end, as usual with an example. Consider the delayed logistic equation (Maynard-smith [1971], Pounder and Rogers [1980], Aronson *et al.*[1980,1982]):

$$F_{\mu}:(x,y) \to (y,\mu)(\ 1 - x)). \qquad (4.5.18)$$

This map has fixed points at

(x,y) = (0,0) and (x,y)= ((μ-1)/μ,(μ-1)/ μ.

We can check that, for μ > 1, (0,0) is a saddle point. the matrix of the map linearized and the other non zero-fixed point is

$$DF\begin{pmatrix} \frac{\mu - 1}{\mu}, & \frac{\mu - 1}{\mu} \end{pmatrix} = \begin{pmatrix} o & 1 \\ 1 - \mu & 1 \end{pmatrix}, \qquad (4.5.19)$$

wihich has eigenvalues

$$\lambda_{1,2} = ½(\ 1 \pm \sqrt{5-4\mu}). \qquad (4.4.20)$$

For $\mu > 5/4$, these eigenvalues are complex conjugate and may be written

$$\lambda, \bar{\lambda} = (\mu - 1)\, e^{\pm ic}, \text{ where } \tan c = \sqrt{4\mu - 5} \, . \quad (4.5.21)$$

It is now easy to check that hypotheses (SH1) and (SH2) of theorem 4.5.2 hold, since, at $\mu = 2$, $\lambda, \bar{\lambda} = e^{\pm i\pi/3}$ are sixth roots of unity, while

$$\frac{d}{d\mu} \left| \lambda(\mu) \right|_{\mu=2} = 1.$$

To compute a from Equation (4.5.17), and hence check (SH3), we set $\mu = 2$ in (4.5.18) and apply the changes of coordinates

$$(\bar{x}, \bar{y}) = (x - ½, y - ½),$$

and

$$\begin{pmatrix} \mu \\ \upsilon \end{pmatrix} = \begin{bmatrix} -1/\sqrt{3} & 2/\sqrt{3} \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} ; \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \begin{bmatrix} 0 & 1 \\ \sqrt{3}/2 & ½ \end{bmatrix} \begin{pmatrix} u \\ \upsilon \end{pmatrix} \quad (4.5.22)$$

which translate the bifurcating equilibrium to the origin, and bring the linear part into normal form. Under these transformations (4.5.18) becomes

$$\begin{pmatrix} u \\ \upsilon \end{pmatrix} \rightarrow \begin{pmatrix} ½ & -\sqrt{3}/2 \\ \sqrt{3}/2 & ½ \end{pmatrix} \begin{pmatrix} u \\ \upsilon \end{pmatrix} - \begin{pmatrix} 2u\upsilon + 2\upsilon^2 \\ 0 \end{pmatrix} \quad (4.5.23)$$

with eigenvalues $\lambda, \lambda = ½ \pm i(\sqrt{3}/2)$.the nonlinear terms are quadratic and we have the following

$$f_{uu} = 0, \; f_{u\upsilon} = -2, \quad f_{\upsilon\upsilon} = -4, \qquad (4.5.24)$$

$$g_{uu} = 0, \quad g_{u\upsilon} = 0, \quad g_{\upsilon\upsilon} = 0.$$

We therefore obtain

$$\xi_{20} = \tfrac{1}{8}[\,4 + 4i] = \tfrac{1}{2} + \tfrac{1}{2},$$
$$\xi_{11} = \tfrac{1}{4}[\,-4 + 0i] = -1, \qquad (4.5.25)$$
$$\xi_{02} = \tfrac{1}{8}[\,4 - 4i] = \tfrac{1}{2} - \tfrac{1}{2},$$
$$\xi_{21} = 0,$$

and substitution into the formula for *a* yields

$$a = \frac{\sqrt{3} - 7}{4} \quad < 0 \qquad (4.5.26)$$

Since $(d/d\mu)(|\lambda(\mu)|)_{\mu=2} = d = 1 > 0$, we deduce form (15) that the bifurcation is supercritical and hence that an attacting invariant closed curve exists, surrounding $(x,y) = (\tfrac{1}{2},\tfrac{1}{2})$ for $\mu > 2$ and $|\mu - 2|$ small.

4.6 **A Nonlinear two-species oscillatory system bifurction and stability analysis**

In this section dealing with the nonlinear bifurcation a nanlysis of two-species oscillatory system consists of three parts. The first part deals with Hopf-bifurcation and limit cycle analysis of the homogeneous system. The second consists of trav-elling wave train solution and its linear stability analysis of the system in presence of diffusion. The last deals with an oscillatory chemical system as an illustrative example.

Introduction: periodicity is an inherent phenomenon in living systems, from the cell cycle, which governs the rate and timing of mitosis(cell division), to the diurnal cycle that result in sleep-wake, to the ebb and flow of populations in their natural environment, life proceeds in a rhythmic and periodic style. Within the nature, several dynamical systems exhibit a large variely of oscillations. The spring-mass system, electrical circuits. Lotka-Volterra predation model system, and so forth , exhibit several types of periodic behaviour. There are some stable periodic behaviours which are not easily disrupted by a perturbation, deterministic or random. These types of situation lead us to believe that pattern is a ubiquitous part of the process of growth of biochemical and metabolic control systems and of ecological systems.

Reaction-diffusion processes play a significant role in the study of pattern formation in different biological and ecological system[6].A large class of nonlinear parabolic partial differential equations are referred to as reaction-diffusion equations[6]. The systems governed by this type of equations are known as reaction-diffusion system. For example, if $u_i(x,t), i = 1,2, \ldots, m$ represents the densities or concentration of several interacting species or chemicals asch of which diffusing with diffusing with their own diffusion coefficients D_i and interacting according to the vector source term f, then the system is governed by [6] ,

$$\frac{\partial u}{} = f + D \nabla^2 u, \qquad 4.6.1)$$

∂t

Where D is a simple diagonal matrix of order m for the case of no cross diffusion. Equation (4.6.1) is referred to as a reaction–diffusion or an interacting population diffusion system [6]. It is believed that rotating and spiral waves are possible solutions of reaction-diffusion equations in appropriate circumstances. Rotating spiral waves have been found by Winfree [6] for the Belousov-zhabotinskii reaction. Kuramoto and Yamada [6] considered a two-species reaction- diffusion system exhibiting limit cycle behaviour. Cohen el al. [6] were the first to demonstrate that rotating spiral wave can be maintained by a reaction-diffusion mechanism alone. They found solutions for the λ - ω system see [6]

$$\frac{\partial u}{\partial t} = \lambda(R)u - \omega(R)\upsilon + \nabla^2 u, \qquad (4.6.2)$$

$$\frac{\partial \upsilon}{\partial t} = \omega(R)u + \lambda(R)\upsilon + \nabla^2 \upsilon,$$

Where λ, ω are given functions of $R = (u^2 + \upsilon^2)^{1/2}$. Stability of traveling waves can often quite difficult to demonstrate analytically. However, some stability results can be obtained, without long and complicated analysis , in the case of the wave train solutions of the λ– ω system 6 . Feroe 6 investigated the systems for which this stability work is to develop a limit or not.

The object of the present work is to develop a limit cycle solution of a general nonlinear two-species model system and then to obtain the criteria of the stability. In section 3, we have tried to find the traveling wave train solution of the above-mentioned problem in presence of diffusive perturbation for both species. We have also performed the linear stability analysis for the traveling wave train solution. As an illustrative example, we have considered a nonlinear reaction-diffusion model equation which governs a certain chemical reaction system introduced by Dreitlein and Somes[6] for which the criterion of linear stability for traveling wave train solution have been tested.

A nonlinear system: Hopf-bifurcation and limit cycley.

We consider a nonlinear system of two interacting species(ecological or chemical) whose concentrations are denoted by $x_1(t)$ and $x_2(t)$ and is governed by the system of equations[6].

$$\frac{dx_1}{dt} = \gamma x_1 - \omega x_2 + (mx_1 - nx_2)(x_1^2 + x_2^2),$$
$$\frac{dx_1}{dt} = \omega x_1 - \gamma x_2 + (nx_1 - mx_2)(x_1^2 + x_2^2), \tag{4.6.3}$$

where γ is a scalar control parameter and ω, m, n are constants. This highly nonlinear planar model is a generalization of various types of nonlinear differential equations governing a variety of physical and chemical systems [6]. Evidently,

$(x_1^*, x_2^*) = (0,0)$ is the fixed point of (4.6.3) for all values of the con-trol parameter γ. Let the matrix $A(\gamma)$ be the linearized matrix of (4.6.3) about. the fixed point(x^*_1, x^*_2), that is,

$$A(\gamma) = \left(\nabla x F(x,\gamma) \right)(x^*) = \begin{pmatrix} \gamma & -\omega \\ \omega & \gamma \end{pmatrix} \quad (4.6.4$$

where

$$x = (x_1, x_2)^T, \; F(x,\gamma) = (F_1, F_2)^T,$$

$$F_1 = \gamma x_1 - \omega x_2 + (mx_1 - nx_2)(x^2_1 + x^2_2), \quad (4.6.5)$$

$$F_2 = \omega x_1 - \gamma x_2 + (nx_1 - mx_2)(x^2_1 + x^2_2),$$

The eigenvalues of the corresponding Jacobian matrix(4.6.4) are $\lambda_1 = \gamma - i\omega$, $\lambda_2 = \gamma + i\omega$ (4.6.6).

From these eigenvalues, we note that (0,0) is a nonhyperbolic fixed point [6] of (4.6.3) when $\gamma = 0$, Further at $(x_1, x_2, \gamma) = (0,0,0)$,

$$\frac{d\lambda_1}{d\gamma} = 1, \quad \frac{d\lambda_2}{d\gamma} = 1, \quad (4.6.7)$$

Hence, all the conditions required for a Hopf-bifurcation are satisfied [6]. Alternatively, the matrix A(0) has purely imaginary eigenvalues $\pm i\omega$ $(\omega \neq 0)$, that is, the conditions [6]

$$TrA(0) = 0, \qquad \det A(0) = \omega^2 > 0 \quad (4.6.8)$$

are satisfied. Also the matrix $B(\gamma)$, defined by

$$A(\gamma) = A(0) + \gamma B(\gamma) \quad (4.6.9)$$

Is such that $TrB(0) = 2 \neq 0$, so there must exist a periodic solution of (4.6.3) for y in some neighborhood of $\gamma = 0$ and x in some neighborhood of x* with approximate period $T = 2\pi/\omega$ for small γ 6 . We can now apply Hopf-bifurcation and limit cycle theorem 6 to find the periodic solution of the system of (4.6.3). Following Murray 6 and without going into the details of calculation, we can show that the periodic solution of (4.6.3) is given by

$$X(\gamma,t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(-\frac{\gamma\,(1+\omega^2)}{2m}\right)^{1/2}\begin{pmatrix} \cos(\omega t+\alpha) \\ \sin(\omega t+\alpha) \end{pmatrix} + 0(\gamma), \quad (4.6.10)$$

where $\alpha = \arctan\,(b_2/b_1)$ is some arbitrary phase angle. From (4.6.10), we see that the amplitude of oscillations depends on the parameters γ, ω, and m. For the existence of the limit cycle, the amplitude should be positive and this requires γ and m to be of opposite sign [6]. Now we consieder the stability of the limit cycle (4.6.10). For this we calculate the Lyapunov – number σ for system (4.6.3).

Following Perko [6], we can calculate the Lyapunov-number σ for system (4.6.3) about the stationary state x* as

$$\sigma = \frac{12\pi}{\omega} m. \quad (4.6.11)$$

If $\sigma \neq 0$, the fixed point (0,0) is a weak focus of multiplicity one. The weak focus will be stable or unstable according to whether $\sigma < 0$, or $\sigma > 0$, that is, according to whether $\omega\, m < 0$ or $\omega\, m > 0$. Hopf-bifurcation occurs at the critical value $\gamma = 0$. if $\sigma < 0$ or $\omega\, m < 0$, the Hopf-bifurcation is supercritical and, on the other hand, if $\sigma > 0$ or $\omega\ m > 0$, the Hopf-bifurcation is subcritical.

Nonlinear reaction-diffusion system: traveling wave trains and linear stability analysis:

We consider the behaviour of the system governed by (4.6.3) in presence of diffusion. We try to find the wave train solutions for the reaction-diffusion system give by

$$\frac{\partial x_1}{\partial t} = \gamma x_1 - \omega x_2 + (m x_1 - n x_2)(x^2_1 + x^2_2) + \frac{\partial^2 x_1}{\partial x^2}$$
$$\frac{\partial x_1}{\partial t} = \omega x_1 - \gamma\, x_2 + (n x_1 - m x_2)(x^2_1 + x^2_2) + \frac{\partial^2 x_1}{\partial x^2} \qquad (4.6.12)$$

For our purpose here, we consider the system with the same rate of diffusion for both species and then it is scaled into a new space variable by using the transformation $x \to x / \sqrt{D}$, where D is the same rate of diffusion for both species. We have shown in the previous section that γ is a bifurcation parameter and when it passes through the value zero, Hopf-bifurcation takes place. We assume the traveling wave train solution of system(4.6.12) in the form

$$V(x,t) = \begin{pmatrix} x_1(x,t) \\ x_2\,(x,t) \end{pmatrix} = V(z), \text{ where } z = \sigma t - kx \qquad (4.6.13)$$

With σ (>0) being the frequency of the wave train, the wave number, and v a periodic function of z with period 2π. Then the wavelength is $\omega = 2\pi/k$ and the wave propagates with the speed $c = \sigma/k$. Substitution of (4.6.13) into(4.6.12) results in a system of ordinary differential equations for v given by

$$K'2V''2 - \sigma V' + f(V) = 0, \qquad (4.6.14)$$

where prime denotes the differentiation of V with respect to the independent variable z. We want to find σ and k so that the last equation has a 2π-periodic solution for v. We can rewrite system (4.6.12) as follows:

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} \lambda(\gamma) & \mu(\gamma) \\ -\mu(\gamma) & \lambda(\gamma) \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{\partial^2}{\partial x^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (4.6.15)$$

Where

$$\gamma^2 = x_2^2 + x_2^2, \quad \lambda(\gamma) = \gamma + mr^2, \quad \mu(r) = -(\omega + nr^2). \qquad (4.6.16)$$

Now changing the system of equations (4.6.15) into polar (r,θ) form, we can write it as follows:

$$\frac{\partial r}{\partial t} = r\,\lambda(r) + r_{xx} - r\theta^2_x$$

$$\frac{\partial \theta}{\partial t} = \mu(r) + \frac{1}{r^2}\frac{\partial}{\partial x}(r^2\theta_x). \qquad (4.6.17)$$

As we are looking for the traveling wave train solutions of the type (4.6.13) in polar form, so we substitute

$$r = \alpha, \qquad \theta = \sigma t - kx \qquad (4.6.18)$$

Into (4.6.17) to get the necessary and sufficient condition for the existence of traveling wave solution. These conditions are obtained after substitution of (4.6.18) into (4.6.17) as

$$\sigma = \mu(\alpha), \qquad k^2 = \lambda(\alpha) \qquad (4.6.19)$$

Considering α as a parameter, the one-parameter, the one-parameter family of traveling wave train solutions of (4.6.15) is given by

train solutions of (4.6.15) is give by

$$x_1 = \alpha \cos\left[\mu(\alpha)t - x\,\lambda^{1/2}(\alpha)\right], \quad x_2 = \alpha \sin\left[\mu(\alpha)t - x\,\lambda^{1/2}(\alpha)\right]$$

(4.6.20)

with wave speed

$$c = \frac{\sigma}{k} = \frac{\mu(\alpha)}{\lambda^{1/2}(\alpha)} \qquad (4.6.21)$$

Such traveling wave trains are of importance, for example, to the target patterns or circular waves generated by the pace-maker nuclei in the Belousov-Zhabotinski reactions [6] .

After finding the traveling wave solution of the λ - μ system described by (4.6.15), we now perform the linear stability analysis of the wave train solution. The simplicity of the plane wave solutions in the polar forms (4.6.17), (4.6.18), (4.6.19), and (4.6.20) gives us the opportunity to do the linear stability analysis. For this linear stability analysis, we consider the perturbations described by

$$r = \alpha + p(x.t), \qquad \theta = \sigma t - kx + \phi(x,t), \qquad (4.6.22)$$

where $|p|$, $|\phi| << 1$. Substituting this relation into (4.6.17) and then linearizing, we get the following equations in terms of the perturbation variables p and ϕ as

$$\frac{\partial p}{\partial t} = \alpha\left(2m\alpha P + 2K\frac{\partial \phi}{\partial x}\right) + \frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial \phi}{\partial t} = -2n\alpha p - \frac{2K}{\alpha}\frac{\partial P}{\partial x} + \frac{\partial^2 \phi}{\partial x^2} \qquad (4.6.23)$$

The conditions satisfied by k and σ under which the solutions of (4.6.23) is to be determined tend to zero as t approaches infinity. As coefficients involved with the system of equations (4.6.23) are constants, we can assume the solution of the system in the Fourier form

$$\begin{pmatrix} p \\ \phi \end{pmatrix} = \begin{pmatrix} p_o \\ \phi_o \end{pmatrix} \exp(st+iqx), \qquad (4.6.24)$$

where s is the growth rate of perturbation, q is the perturbation wave number, and p_o and ϕ_o are constants. The stability of the linearized system demands that Re(s)<0. Substituting(4.6.24) into (4.6.23), we get.

$$\begin{pmatrix} s + q2 \ - 2m\alpha^2 & -2ik\alpha q \\ 2n\alpha + 2ik \ \dfrac{q}{\alpha} & s + q^2 \end{pmatrix} \begin{pmatrix} p_o \\ \phi_o \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (4.6.25)$$

As we are searching for nontrivial solution, then we are not interested in the solution p_o, $\phi_o = 0$, so we must have the determinant value of the 2× 2 matrix involved with (4.6.25) equal to zero , which is a quadratic equation in s. If we denote the roots of the quadratic equation by s_1 and s_2, then expressions for the roots are given by.

$$s_1, s_2 = - q^2 + \alpha^2 m \pm (m^2\alpha^4 + 4k^2q^{2-}2iknq\alpha^2)^{1/2} \qquad (4.6.26)$$

Depending upon the parameters of the system which are nivolved in (4.6.26), the real parts of both s_1 and s_2 or of either s_1 or s_2 may be positive, and then the plane wave solutions will be linearly unstable. As and q are perturbation from the plane wave solutions (4.6.18), then the perturbation wave number q = 0 leads to the fast that $s_1 = 2\alpha^2 m$ and $s_2 = 0$. The later corresponds to the neutral stability or instability depending upon the condition satisfied by m, and is given by m < 0 m or m . 0, respectively. For positive perturbation in wave number, that is , $q>0$, the maximum real parts of the roots come from s_1 and this leads to the necessary and sufficient condition for linear stability, namely, $Re(s_1) < 0$. From (4.6.26), after some calculations involving complex variable algebra, we find that

$$Re(s_1(q)) = -q^2 + \alpha^2 m + \frac{1}{\sqrt{2}}\left[(m\alpha^2)^2 + 4k^2q^2 + \left[\left((m\alpha^2)^2 + 4k^2q^2\right)^2 + 4(knq\alpha^2)^2\right]^{1/2}\right]. \qquad (4.6.27)$$

From the above relation, we get

$$Re(s_1(0)) = \alpha^2 m + |\alpha^2 m|, \qquad (4.6.28)$$

$$\left(\frac{dRe\, s_1}{dq^2}\right)_{q=0} = -1 + \frac{4k^2(1+n^2/m^2)}{2\alpha^2\,|m|} \qquad (4.6.29)$$

Relation (4.6.28) states that $Re\ s_1(0) = 2\alpha^2 m$ for $m > 0$ and $Re\ s_1(0) = 0$ for $m < 0$. Thus for small enough q^2, $Re\ s_1(q) < 0$ if and only if the last derivative $(dRes_1/dq^2)_{q=0} < 0$. For $m < 0$, the relation (4.6.29) gives the condition as

$$4k^2\left(1 + \frac{n^2}{m^2}\right) + 2\alpha^2 m \le 0, \qquad (4.6.30)$$

whereas, for $m > 0$, $Re\ s_1(0) > 0$ and consequently the traveling wave train solution of system (4.6.15) unstable.

Travelling wave in an oscillatory chemical system linear stability analysis:

We now consider a model chemical reaction as an illustrative example of the general nonlinear system whose different characteristic features we have discussed in the previous section. This model for oscillatory chemical kinetic system was discussed and analyzed by Dreitlein and somes 6 . Here we analyze traveling wave solution and stability. The model is described by the system of nonlinear equation as follows:

$$\frac{\partial x_1}{\partial t} = EX1 + 2X^2 - X_1(X^2_1 + X^2_2)\frac{\partial^2 x_1}{\partial x^2} \qquad (4.6.31)$$

$$\frac{\partial x_1}{\partial t} = -2X1 + EX_2 - X2(X^2 1 + X^2_2) + \frac{\partial^2 x_2}{\partial x^2}$$

Comparing this equation with (4.6.12), we find that $\gamma = E$, $\omega = -2$,$m = -1$, and $n = 0$. The system of equations (4.6.31) can be written as a λ- ω model system as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda(r) & \mu(r) \\ -\mu(r) & \lambda(r) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{\partial^2}{\partial x^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (4.6.32)$$

where

$$r^2 = x_2^2 + x_2^2, \qquad \lambda(r) = E - r^2, \quad \mu(r) = 2, \qquad (4.6.33)$$

Now $r = r_o = \sqrt{E}$ is an isolated zero of $\lambda(r)$ and, consequently, $\lambda'(r_o) = -2\sqrt{E} < 0$, $\mu(r_o) = -2 \neq 0$. This leads to the conclusion that the spatially homogeneous system has a limit cycle solution[6]. Changing the variables from (x_1 , x_2) to the polar variable (r, θ) and using (4.6.17) the system of equations (4.6.31) becomes.

$$\frac{\partial r}{\partial t} = r\,(E - r^2) + \frac{\partial^2 r}{\partial x^2} - r\left(\frac{\partial \theta}{\partial x}\right)^2,$$

$$\frac{\partial\theta}{\partial t} = 2 + \frac{1}{r^2}\frac{\partial}{\partial x} - r(\frac{\partial\theta}{\partial x}), \quad (4.6.34)$$

As $r_o = \sqrt{E} > 0$ and $\lambda' (\sqrt{E}\ ) < 0$, then the asymptotically stable limit cycle solution of kinetic system is given by

$$r = \sqrt{E}\ ,\ \theta = \theta_o + 2t, \quad (4.6.35)$$

where θ_0 is some arbitrary phase. Next, we look for travelling wave solution of the form (4.6.14) of the system governed by the system of differential equation (4.6.31). Substituting $r = \alpha, \theta = \sigma t - kx$ in (4.6.34), we can obtain the necessary and sufficient conditions for the solution of the above- mentioned type. The conditions are

$$\sigma = 2\ ,\ k^2 = E - \alpha^2\ , \quad (4.6.36)$$

The one- parameter family of travelling wave train solutions of (4.6.34) or, equivalently, for (4.6.31), is given by

$$x_1 = \alpha\cos\,[2t - x(E - \alpha^2)^{1/2}\,], \quad x_2 = \alpha^2\sin[2t - x(E - \alpha^2)^{1/2}\,], \quad (4.6.37)$$

with α as the convenient parameter. As the parameter α approaches the value $r_0 = \sqrt{E}$, the wave number of the plane waves tends to zero and this indicates the existence of travelling plane wave train solutions near the limit cycle. The system governed by (4.6.31) has a steady state at (0, 0) which is stable for $E > 0$ and unstable for $E < 0$. Note that $E = E_c = 0$ is the bifurcation value of the system.At the critical value $Ec = 0$, the eigenvalues of the linearized system about the steady state are

$\pm$ 2i. This satisfies the requirements of Hopf- bifurcation that we have discussed.

Our next task is to investigate the linear stability of the wave train solution that we have discussed for system (4.6.12). In a similar manner as that we have adopted before, we will deal with the polar form (4.6.34) and the perturbations will be the form

$$r = \propto + \rho(x+t), \quad \theta = \sigma t - kx + \phi(x+t) \qquad (4.6.38)$$

where $|\rho|, |\phi| \ll 1$. Substituting this relation into (4.6.34) and linearizing, we get

$$\frac{\partial \rho}{\partial t} = \propto [-2 \propto \rho + 2k \frac{\partial \phi}{\partial x}] + \frac{\partial^2 \rho}{\partial x^2}, \qquad (4.6.39)$$

$$\frac{\partial \rho}{\partial t} = \frac{2k}{\alpha} \frac{\partial \rho}{\partial x} + \frac{\partial^2 \phi}{\partial x^2},$$

The coefficients involved in the linearlized system (4.6.39) are all constants and this situation enable us to take the solution of this system in the form

$$\begin{pmatrix} \rho \\ \phi \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \phi_0 \end{pmatrix} \exp(st + iqx), \qquad (4.6.40)$$

where ρ_0, ϕ_0 are constants and q, s play the same role as that involved in (4.6.24). Substituting (4.6.40) into the system of equations (4.6.39), we get relations of the form (4.6.25) as follows

$$\begin{pmatrix} s + q^2 + 2\alpha^2 & -2ik\alpha q \\ 2ik \frac{q}{\alpha} & s+q2 \end{pmatrix} \begin{pmatrix} p_o \\ \phi_o \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (4.6.41)$$

As we are interested in the nontrivial solution of the system (4.6.41), then we must have the determinant value of the coefficient matrix involved in (4.6.41) equal to zero. This determinant value of the above- mentioned matrix equal to zero gives a quadratic equation for the variables s. The stability of the linearized system requires that the roots of the quadratic equation in s have negative real parts. If we denote the two roots of the quadratic equation by s_1, s_2, then the expressions for them can be given as follows (using the expression (4.6.26):

$$S_1,S_2=q^2-\alpha^2 \pm\left(\alpha^4+4k^2q^2\right)1/2 \qquad (4.6.39)$$

From the above relation we get $s_1(0)$, $s_2(0) = -2\propto^2$. Thus, for vanishingly small perturbation in the wave number, the linearlized system exhibits a neutral stability. Now, for small perturbation, both roots $s_1$and s_2 will be negative if and only if

$$2k^2-\alpha^2 < 0 \qquad (4.6.43)$$

This is the condition for stability of the linearized system and it is independent of the parameter E involved with system (4.6.31).

Conclusion:

The bifurcation theory plays a significant role in the behavior of nonlinear systems. The bifurcating behavior for a nonlinear system is a self-developed phenomena for the deterministic system [6] . The first problem in this section is the study of an interacting homogeneous population system governed by the nonlinear system of differential equations(4.6.3). The Hopf-bifurcation analysis of the system leads to an unstable or a stable limit cycle according to whether the system leads to an unstable or a stable limit cycle according to whether the bifurcation parameter is negative or positive. The limit cycle solution (4.6.10) of the nonlinear system (4.6.3) shows uninteresting characteristic that the existence of the limit cycle solution depends upon the sign of the parameter γ and m involved with the system. The existence of limit cycle solution of system(4.6.3) demands $m\gamma < 0$. However, when $m = 0$, although the conditions for Hopf-bifurcation are satisfied, there are no periodic orbits in the vicinity of the bifurcation point.

The next problem is concerned with the study of traveling wave train solution of the diffusive nonlinear dynamic system and the linear stability criteria of this wave train solution. Equation (4.6.20) represents the one- parameter family of wave train solution for system (4.6.12), where $\propto$ is the arbitrary parameter. If $r = r_o$ is an isolated zero of $\lambda(r)$, (given by (4.6.16)), then the limiting approach $r_o \longrightarrow \propto$ gives the small amplitude travlling wave train solution near the limit cycle arising from Hopf-bifurcation. Kopell and Howard [6] showed how to do this in general. Feroe [6] has discussed the difficulties in the stability of traveling wave solution for excitable FHN waves. However, we are able to find this criteria for system (4.6.12) given by (4.6.30) without long and complicated calculations due to the simplicity of traveling wave train [6] .

As an illustrative example, we have considered the chemical reaction-diffusion model introduced by Dreitlein and Somes [6].The traveling wave train solution of the model system(4.6.31)given by (4.6.37) is quite similar to the result introduced by Dreitlin and Somes. condition (4.6.43) is the linear stability condition of the wave train solution(4.6.37) and it should be noted that (4.6.43) does not contain the parameter E of the model system (4.6.31). The stability condition (4,6.43) may be derived from (4.6.30) by substituting the values of m and n for the model system (4.6.31).

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