

Nile valley university

college of graduate studies

On the complex analysis :

**The Analytic Continuation
Problems**

By :

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requirement for M.Sc Degree in mathematics**

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Dedication

To my mother

Acknowledgement

First , my special prais and thanks to Allah .

I wish to express sincere thanks and grateute to my supervior Dr.Adam Abd Allah Abbaker who suggest the title of this study and for this invaluable help , suggestions and encouragement during the course of this study

I also like to thank the staff of Nile Valley university for the chance they afforded us to enjoy such a nice program .

Abstract

Analytic continuation provides a way to extending the domain over which a complex function is defined.

We can find an analytic continuation by finding Taylor series to the given function $f_0(z) = \sum_{n=0}^{\infty} (z - z_0)^n$ which is convergence on $|z - z_0| < R$ and its centre in z_0 in \mathbb{C}_0 .

If z_1 satisfies $|z_1 - z_0| < R$ we can write f_0 in a power series

$$f_1(z) = \sum_{n=0}^{\infty} b_n (z - z_1)^n \quad b_n = f_0^{(n)}(z_1) / n!$$

Mondromy theorem is an important result about analytic continuation of a complex analytic function to a larger set.

Analytic continuation has applications in many sciences, the study gives some physical and biological.

ملخص الدراسة

الامتداد التحليلي يمكننا من توسيع تعريف الدالة لتكون تحليلية على منطقة اكبر .

يمكن الحصول على امتداد تحليلي بايجاد متسلسلة تايلور للدالة المعطاة $f(z) = \sum_{n=0}^{\infty} (z-z_0)^n$

و المتقاربة على القرص $|z-z_0| < R$ الذي مركزه z_0 في C_0

اذا كانت z_1 تحقق $|z_1-z_0| < R_0$

فبامكاننا كتابة f_0 على شكل متسلسلة قوى $f_1 = \sum_{n=0}^{\infty} b_n (z-z_1)^n$ $b_n = f^n(z_n)/n!$

نظرية الموندرمي نتيجة مهمة للامتداد التحليلي .

الامتداد التحليلي له تطبيقات في العلوم الاخرى تناولت الدراسة بعض تطبيقاته في الفيزياء

والبيولوجيا .

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Introduction

The complex analysis has important roles in modern physics , and great variety of problems in physics – both conceptual and technical can be explored by using it . from there the searcher chose the analytic continuation in complex analytic to be his research .

The research consists of four chapters :

Chapter 1 speaks a bout complex analysis and it considered as a door to the rest of the following chapters ,

Chapter 2 speaks a bout analytic continuation in analytic functions and consists of four sections , the first section is definition of analytic continuation , the second section is analytic functions defined of real variables , the third section is polar coordinates and the fourth section is analytic functions defined in terms of complex plane .

Chapter 3 speaks a bout analytic continuation of summation and analytic continuation a long a curve and it consists of four sections , the first section defines analytic continuation of sum , the second section is a bout the power series , section three is about analytic continuation a long a curve and section four is about Mondormy theorem which is an important result about analytic continuation of complex analytic function to a larger set .

Chapter 4 speaks about the applications of analytic continuation and its consists of four sections , section one is about the applications in complex analytic functions , section two is about application of numerical analytic continuation in traction boundary value problems for the half plane , section three speaks about study of quantum mechanical relaxation process and section four speaks about fractal theory or geometry.

Research plan

Title : On the complex Analysis : The Analytic continuation problems .

The Research Method

Scientific method

Aims of the research

- 1- Presenting the importance of analytic continuation for solving the problems in complex analysis .
- 2- Presenting the applications of analytic continuation .

Importance of the research

The importance of this research comes from the importance of complex analysis and its applications .

The Questions of the research

Some times the express $f_0(z)$. such as in infinite series or integration present and analytic function has meaning

The main question is :

Is there process for extending the definition of the analytic function to be analytic on wide region

Consist of these question :

- 1- Can we find a function $f_1(z)$ analytic on a region G_1 . for all values of z in $G_0 \cap G_1$ and can we generalize that ?
- 2- Is there any process to find analytic continuation by finding Taylor series to the function

$$F_0(z) = \sum_{n=0}^{\infty} (z-z_0)^n$$

which is converges on $|z - z_0| < R_0$ which its center in the point z_0 in G_0

- 3- What is Mondormy theorem and what its importance
- 4- What is applications of the analytic continuation

Assumption of the research

1. We can find a function $f_1(z)$ analytic on a region G_1 intersect with G_0 and it should be $f_0(z) = f_1(z)$ for all values of z in $G_0 \cap G_1$ and we can generalize that to $G_0 \cup G_1$
- 2- We can find analytic continuation by finding Taylor series .
- 3- If γ, γ' are two separated arcs except the two ending point and it should not be found isolated points on or inside the closed curve $\gamma - \gamma'$ then the two curves yield the same result at there common end point .
- 4- Analytic continuation has an important applications



chapter 1

Complex analysis :

1-1 History

Complex analysis is one of the classical branches in mathematics with its roots in the 19th century and some even before. Important names are Euler, Gauss, Riemann, Cauchy, Weierstrass, and many more in the 20th century. Traditionally, complex analysis, in particular the theory of conformal mappings, has many physical applications and is also used throughout analytic number theory. In modern times, it became very popular through a new boost of complex dynamics and the pictures of fractals produced by iterating holomorphic functions, the most popular being the Mandelbrot set. Another important application of complex analysis today is in string theory which is a conformally invariant quantum field theory.[2]

1-2 definition :

Complex analysis: traditionally known as the theory of functions of a complex variable, is the branch of mathematics investigating functions of complex numbers. It is useful in many branches of mathematics, including number theory and applied mathematics, and in physics.

Complex analysis is particularly concerned with the analytic functions of complex variables (or, more generally, meromorphic functions). Because the separable real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in physics. [3]

1-3 Complex functions

A complex function is a function in which the independent variable and the dependent variable are both complex numbers. More precisely, a complex function is a function whose domain Ω is a subset of the complex plane and whose range is also a subset of the complex plane.

For any complex function, both the independent variable and the dependent variable may be separated into real and imaginary parts:

$$z = x + iy \quad \text{and}$$
$$w = f(z) = u(z) + iv(z)$$

where $x, y \in \mathbb{R}$ and $u(z), v(z)$ are real-valued functions.

In other words, the components of the function $f(z)$,

$$u = u(x, y) \quad \text{and}$$
$$v = v(x, y),$$

can be interpreted as real valued functions of the two real variables, x and y .

The basic concepts of complex analysis are often introduced by extending the elementary real functions (e.g., exponentials, logarithms, and trigonometric functions) into the complex domain. [1]

1-4 Derivatives and the Cauchy–Riemann equations:

Just as in real analysis, a complex function $w = f(z)$ may have a derivative at a particular point in its domain Ω . In fact, the definition of the derivative

$$f'(z) = \frac{dw}{dz} = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

is analogous to the real case but with one very important difference. In real analysis, the limit can only be approached by moving along the one-dimensional number line. In complex analysis, the limit can be approached from any direction in the two-dimensional complex plane, and for the derivative to exist, the limiting value must be the same no matter what the direction of approach of h to 0.

(The claim that "in real analysis, the limit can only be approached by moving along the one dimensional number line" should not be confused with directional derivatives. It may be explained that in directional derivatives, one still moves along the one dimensional x line but it can be in "discrete" units; that is, if one follows the $y = x^2$ curve, that does not mean that one is moving on the plane (instead of the one dimensional x line) but means that one is approaching in steps of discrete units.)

If this limit, the derivative, exists for every point z in Ω , then $f(z)$ is said to be differentiable on Ω . It can be shown that any differentiable $f(z)$ is analytic. This is a much more powerful result than the analogous theorem that can be proved for real-valued functions of real numbers. In the calculus of real numbers, we can construct a function $f(x)$ that has a first derivative everywhere, but for which the second derivative does not exist at one or more points in the function's domain. But in the complex plane, if a function $f(z)$ is differentiable in a neighborhood it must also be infinitely differentiable in that neighborhood.[6]

By applying the methods of vector calculus to compute the partial derivatives of the two real functions $u(x, y)$ and $v(x, y)$ into which $f(z)$ can be decomposed, and by considering two paths leading to a point z in Ω , it can be shown that the existence of derivative implies

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Equating the real and imaginary parts of these two expressions we obtain the traditional formulation of the Cauchy–Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ or, in another common notation,}$$

$$u_x = v_y \quad u_y = -v_x.$$

By differentiating this system of two partial differential equations first with respect to x, and then with respect to y, we can easily show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \text{ or, in another common}$$

$$\text{notation, } u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0.$$

In other words, the real and imaginary parts of a differentiable function of a complex variable are harmonic functions because they satisfy Laplace's equation [6]

1-5 Holomorphic functions:

Holomorphic functions are complex functions defined on an open subset of the complex plane which are differentiable. Complex differentiability has much stronger consequences than usual (real) differentiability. For instance, holomorphic functions are infinitely differentiable, a fact that is far from true for real differentiable functions. Most elementary functions, including the exponential function, the trigonometric functions, and all polynomial functions, are holomorphic.

One central tool in complex analysis is the line integral. The integral around a closed path of a function which is holomorphic everywhere inside the area bounded by the closed path is always zero; this is the Cauchy integral theorem. The values of a holomorphic function inside a disk can be computed by a certain path integral on the disk's boundary. Path integrals in the complex plane are often used to determine complicated real integrals, and here the theory of residues among others is useful. If a function has a pole or singularity at some point, that is, at that point its values do not have a finite value, then one can compute the function's residue at that pole, and these residues can be used to compute path integrals involving the function; this is the content of the powerful residue theorem. The remarkable behavior of holomorphic functions near essential singularities is described by Picard's Theorem. Functions which have only poles but no essential singularities are called meromorphic. Laurent series are similar to Taylor series but can be used to study the behavior of functions near singularities.

A bounded function which is holomorphic in the entire complex plane must be constant; this is Liouville's theorem. It can be used to provide a natural and short proof for the fundamental theorem of algebra which states that the field of complex numbers is algebraically closed. [2]

An important property of holomorphic functions is that if a function is holomorphic throughout a simply connected domain then its values are fully determined by its values on any smaller subdomain. The function on the larger domain is said to be analytically continued from its values on the smaller domain. This allows the extension of the definition of functions such as the Riemann zeta function which are initially defined in terms of infinite sums that converge only on limited domains to almost the entire complex plane. Sometimes, as in the case of the natural logarithm, it is impossible to analytically continue a holomorphic function to a non-simply connected

domain in the complex plane but it is possible to extend it to a holomorphic function on a closely related surface known as a Riemann surface.

All this refers to complex analysis in one variable. There is also a very rich theory of complex analysis in more than one complex dimension where the analytic properties such as power series expansion still remain true whereas most of the geometric properties of holomorphic functions in one complex dimension are no longer true. The Riemann mapping theorem about the conformal relationship of certain domains in the complex plane, which may be the most important result in the one-dimensional theory, fails dramatically in higher dimensions. It is also applied in many subjects through out engineering particularly in power engineering. [2]

chapter 2

Analytic function

In this chapter we will study analytic function and how we can solve the problems of analytic function using analytic continuation

2-1 Definition :

In complex analysis a branch of mathematics analytic continuation is a technique to extend the domain of definition of a given analytic function analytic continuation often succeeds in defining further values of a function , for example in a new region where an infinite series representation in terms of which it is initially defined becomes divergent. [3]

Suppose there is a function, $f_1(z)$ that is analytic in the domain D_1 and another analytic function, $f_2(z)$ that is analytic in the domain D_2 .

If the two domains overlap and $f_1(z) = f_2(z)$ in the overlap region $D_1 \cap D_2$, then $f_2(z)$ is called an analytic continuation of $f_1(z)$. This is an appropriate name since $f_2(z)$ continues the definition of $f_1(z)$ outside of its original domain of definition D_1 . We can define a function $f(z)$ that is analytic in the union of the domains $D_1 \cup D_2$. On the domain D_1 we have $f(z) = f_1(z)$ and $f(z) = f_2(z)$ on D_2 . $f_1(z)$ and $f_2(z)$ are called function elements . There is an analytic continuation even if the two domains only share an arc and not a two dimensional region. With more overlapping domains D_3 , D_4, \dots we could perhaps extend $f_1(z)$ to more of the complex plane. Sometimes it is impossible to extend a function beyond the boundary of a domain. This is known as a natural boundary

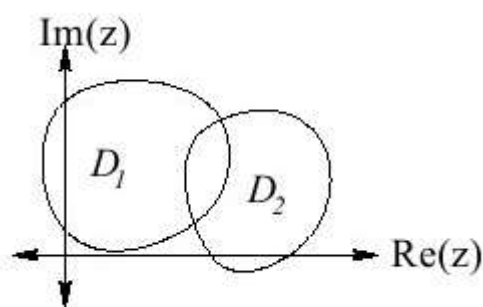


Figure 2.1:

if a function $f_1(z)$ is analytically continued to a domain D_n along two different paths

then the two analytic continuations are identical as long as the paths do not enclose a branch point of the function this is the uniqueness theorem of analytic continuation .

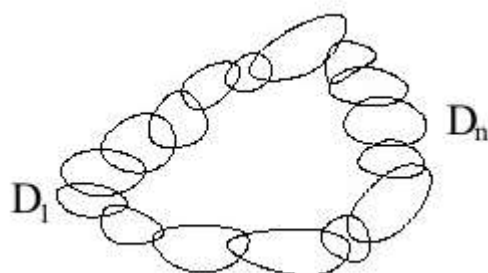


Figure 2.2:

Consider an analytic function $f(z)$ defined in the domain D . Suppose that $f(z) = 0$ on the arc AB , Then $f(z) = 0$ in all of D Consider a point ζ on AB . The Taylor series expansion of $f(z)$ about the point $z = \zeta$ converges in a circle C at

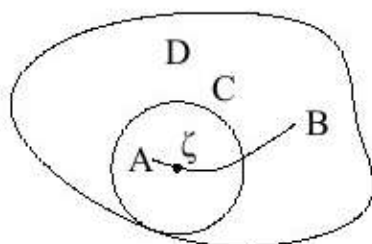


Figure 2.3

least up to the boundary of D . The derivative of $f(z)$ at the point $z = \zeta$ is

$$f'(\zeta) = \lim_{\Delta z \rightarrow 0} \frac{f(\zeta + \Delta z) - f(\zeta)}{\Delta z}$$

If Δz is in the direction of the arc, then $f(\zeta)$ vanishes as well as all higher derivatives, $f''(\zeta) = f'''(\zeta) = f^{(4)}(\zeta) = \dots = 0$. Thus we see that $f(z) = 0$ inside C .

By taking Taylor series expansions about points on AB or inside of C we see that $f(z) = 0$ in D .

Result 2.1 Let $f_1(z)$ and $f_2(z)$ be analytic functions defined in D. If $f_1(z) = f_2(z)$ for the points in a region or on an arc in D, then $f_1(z) = f_2(z)$ for all points in D.

To prove Result 2.1.1, we define the analytic function $g(z) = f_1(z) - f_2(z)$. Since $g(z)$ vanishes in the region or on the arc, then $g(z) = 0$ and hence $f_1(z) = f_2(z)$ for all points in D.

Result 2.2 Consider analytic functions $f_1(z)$ and $f_2(z)$ defined on the domains D_1 and D_2 respectively. Suppose that $D_1 \cap D_2$ is a region or an arc and that $f_1(z) = f_2(z)$ for all points $\in D_1 \cap D_2$. Then the function

$$f(z) = \begin{cases} f_1(z) & \text{for } z \in D_1, \\ f_2(z) & \text{for } z \in D_2, \end{cases}$$

is analytic in $D_1 \cup D_2$ (1)

2-2 Analytic Functions Defined in Terms of Real Variables

Result 2.3. An analytic function, $u(x, y) + iv(x, y)$ can be written in terms of a function of a complex variable, $f(z) = u(x, y) + iv(x, y)$ [1]

Example.2.2.1

$$f(z) = \cosh y \sin x (x e^x \cos y - y e^x \sin y) - \cos x \sinh y (y e^x \cos y + x e^x \sin y) + i [\cosh y \sin x (y e^x \cos y + x e^x \sin y) + \cos x \sinh y (x e^x \cos y - y e^x \sin y)]$$

is an analytic function. Express $f(z)$ in terms of z . On the real line, $y = 0$, $f(z)$ is

$$f(z = x) = x e^x \sin x$$

(Recall that $\cos(0) = \cosh(0) = 1$ and $\sin(0) = \sinh(0) = 0$)

The analytic continuation of $f(z)$ into the complex plane is:

$$f(z) = z e^z \sin z.$$

Alternatively, for $x = 0$ we have

$$f(z = iy) = y \sinh y (\cos y - i \sin y)$$

The analytic continuation from the imaginary axis to the complex plane is

$$\begin{aligned} f(z) &= -iz \sinh(-iz) (\cos(-iz) - i \sin(-iz)) \\ &= iz \sinh(iz) (\cos(iz) + i \sin(iz)) \\ &= z \sin z e^z. \end{aligned}$$

Example.2.2.2

Consider $u = e^{-x}(x \sin y - y \cos y)$ Find v such that $f(z) = u + iv$ is analytic. From the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = e^{-x} \cos y - x e^{-x} \cos y - y e^{-x} \sin y \end{aligned}$$

Integrate the first equation with respect to y .

$$\begin{aligned} v &= -e^{-x} \cos y + x e^{-x} \cos y + e^{-x}(y \sin y + \cos y) + F(x) \\ &= y e^{-x} \sin y + x e^{-x} \cos y + F(x) \end{aligned}$$

$F(x)$ is an arbitrary function of x . Substitute this expression for v into the equation for $\partial v / \partial x$ [1]

$$-y e^{-x} \sin y - x e^{-x} \cos y + e^{-x} \cos y + F'(x) = -y e^{-x} \sin y - x e^{-x} \cos y + e^{-x} \cos y$$

Thus $F'(x) = 0$ and $F(x) = c$.

$$v = e^{-x}(y \sin y + x \cos y) + c$$

Example 2.2.3

Consider

$$f(z) = u(x, y) + iv(x, y).$$

Show that

$$f'(z) = u_x(z, 0) - iv_y(z, 0).$$

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= u_x - iv_y \end{aligned}$$

$f'(z)$ is an analytic function. On the real axis, $z = x$, $f'(z)$ is

$$f'(z = x) = u_x(x, 0) - iv_y(x, 0)$$

Now $f'(z = x)$ is defined on the real line. An analytic continuation of $f'(z = x)$ into the complex plane is

$$f'(z) = u_x(z, 0) - iv_y(z, 0).$$

Example 2.2.4.

Again consider the problem of finding $f(z)$ given that $u(x, y) = e^{-x}(x \sin y - y \cos y)$

Now we can use the result of the previous example to do this problem [1]

$$\begin{aligned} u_x(x, y) &= \frac{\partial u}{\partial x} = e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y \\ u_y(x, y) &= \frac{\partial u}{\partial y} = x e^{-x} \cos y + y e^{-x} \sin y - e^{-x} \cos y \end{aligned}$$

$$\begin{aligned} f'(z) &= u_x(z, 0) - iv_y(z, 0) \\ &= 0 - i(z e^{-z} - e^{-z}) \\ &= i(-z e^{-z} + e^{-z}) \end{aligned}$$

Integration yields the result

$$f(z) = iz e^{-z} + c$$

Example 2.2.5

Find $f(z)$ given that

$$u(x, y) = \cos x \cosh^2 y \sin x + \cos x \sin x \sinh^2 y$$

$$v(x, y) = \cos^2 x \cosh y \sinh y - \cosh y \sin^2 x \sinh y$$

$$f(z) = u(x, y) + iv(x, y)$$

is an analytic function. On the real line, $f(z)$ is

$$f(z = x) = u(x, 0) + iv(x, 0)$$

$$= \cos x \cosh^2 0 \sin x + \cos x \sin x \sinh^2 0 + i(\cos^2 x \cosh 0 \sinh 0 - \cosh 0 \sin^2 x \sinh 0)$$

$$= \cos x \sin x$$

Now we know the definition of $f(z)$ on the real line. We would like to find an analytic continuation of $f(z)$ into the complex plane. An obvious choice for $f(z)$ is

$$f(z) = \cos z \sin z$$

Using trig identities we can write this as

$$f(z) = \frac{\sin(2z)}{2}.$$

Example 2.2.6

Find $f(z)$ given only that

$$u(x, y) = \cos x \cosh^2 y \sin x + \cos x \sin x \sinh^2 y.$$

Recall that

$$f'(z) = u_x + iv_x$$

$$= u_x - vu_y$$

Differentiating $u(x, y)$,

$$u_x = \cos^2 x \cosh^2 y - \cosh^2 y \sin^2 x + \cos^2 x \sinh^2 y - \sin^2 x \sinh^2 y$$

$$u_y = 4 \cos x \cosh y \sin x \sinh y$$

$f'(z)$ is an analytic function. On the real axis, $f'(z)$ is

$$f'(z = x) = \cos^2 x - \sin^2 x$$

Using trig identities we can write this as

$$f'(z = x) = \cos(2x)$$

Now we find an analytic continuation of $f'(z = x)$ into the complex plane

$$f'(z) = \cos(2z)$$

Integration yields the result

$$f(z) = \frac{\sin(2z)}{2} + c$$

2.3 Polar Coordinates

Example. 2.3.7

$$u(r, \theta) = r(\log r \cos \theta - \theta \sin \theta)$$

the real part of an analytic function?

The laplacian in polar coordinates is

$$\Delta \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}.$$

We calculate the partial derivative of u

$$\begin{aligned}\frac{\partial u}{\partial r} &= \cos \theta + \log r \cos \theta - \theta \sin \theta \\ r \frac{\partial u}{\partial r} &= r \cos \theta + r \log r \cos \theta - r \theta \sin \theta \\ \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) &= 2 \cos \theta + \log r \cos \theta - \theta \sin \theta \\ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) &= \frac{1}{r} (2 \cos \theta + \log r \cos \theta - \theta \sin \theta)\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= -r (\theta \cos \theta + \sin \theta + \log r \sin \theta) \\ \frac{\partial^2 u}{\partial \theta^2} &= r (-2 \cos \theta - \log r \cos \theta + \theta \sin \theta) \\ \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= \frac{1}{r} (-2 \cos \theta - \log r \cos \theta + \theta \sin \theta)\end{aligned}$$

From the above we see that

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Therefore u is harmonic and is the real part of same analytic function.[1]

Example 2.3.8

Find an analytic function $f(z)$ whose real part is

$$u(r, \theta) = r (\log r \cos \theta - \theta \sin \theta)$$

Let $f(z) = u(r, \theta) + iv(r, \theta)$. The Cauchy-Riemann equations are

$$u_r = \frac{v_\theta}{r}, \quad u_\theta = -rv_r.$$

Using the partial derivatives in the above example, we obtain two partial differential equations for $v(r, \theta)$

$$\begin{aligned}v_r &= -\frac{u_\theta}{r} = \theta \cos \theta + \sin \theta + \log r \sin \theta \\ v_\theta &= ru_r = r (\cos \theta + \log r \cos \theta - \theta \sin \theta)\end{aligned}$$

Integrating the equation for v_θ yields

$$v = r(\theta \cos \theta + \log r \sin \theta) + F(r)$$

where $F(r)$ is a constant of integration

substituting our expression for v_r into the equation for v_r yields

$$\theta \cos \theta + \log r \sin \theta + \sin \theta + F'(r) = \theta \cos \theta + \sin \theta + \log r \sin \theta$$

$$\theta F'(r) = 0$$

$$F(r) = \text{const}$$

Thus we see that

$$f(z) = u + iv$$

$$= r(\log r \cos \theta - \theta \sin \theta) + ir(\theta \cos \theta + \log r \sin \theta) + \text{const}$$

$f(z)$ is an analytic function. On the line $\theta = 0$, $f(z)$ is

$$f(z = r) = r(\log r) + ir(0) + \text{const}$$

$$= r \log r + \text{const}$$

The analytic continuation into the complex plane is

$$f(z) = z \log z + \text{const}$$

Example.2.3.9

Find the formula in polar coordinates that is analogous to

$$f'(z) = u_x(z, 0) - iu_y(z, 0).$$

We know that

$$\frac{df}{dz} = e^{-i\theta} \frac{\partial f}{\partial r}.$$

If $f(z) = u(r, \theta) + iv(r, \theta)$ then

$$\frac{df}{dz} = e^{-i\theta} (u_r + iv_r)$$

From the Cauchy-Riemann equations, we have $v_r = -u_\theta / r$

$$\frac{df}{dz} = e^{-i\theta} \left(u_r - i \frac{u_\theta}{r} \right)$$

$f'(z)$ is an analytic function. On the line $\theta = 0$, $f'(z)$ is

$$f'(z = r) = u_r(r, 0) - i \frac{u_\theta(r, 0)}{r}$$

The analytic continuation of $f'(z)$ into the complex plane is

$$f'(z) = u_r(z, 0) - \frac{i}{r} u_\theta(z, 0). \quad [1]$$

Example.2.3.10

Find an analytic function $f(z)$ whose real part is

$$u(r, \theta) = r (\log r \cos \theta - \theta \sin \theta)$$

$$u_r(r, \theta) = (\log r \cos \theta - \theta \sin \theta) + \cos \theta$$

$$u_\theta(r, \theta) = r (-\log r \sin \theta - \sin \theta - \theta \cos \theta)$$

$$f'(z) = u_r(z, 0) - \frac{i}{r} u_\theta(z, 0)$$

$$= \log z + 1$$

Integrating $f_0(z)$ yields

$$f(z) = z \log z + ic.$$

2.4 Analytic Functions Defined in Terms of Their Real or Imaginary Parts

Consider an analytic function: $f(z) = u(x, y) + iv(x, y)$. We differentiate this expression.

$$f'(z) = u_x(x, y) + iv_x(x, y)$$

We apply the Cauchy-Riemann equation $v_x = -u_y$

$$f'(z) = u_x(x, y) - iv_y(x, y).$$

Now consider the function of a complex variable $g(\zeta)$:

$$g(\zeta) = u_x(x, \zeta) - iv_y(x, \zeta) = u_x(x, \xi + i\psi) - iv_y(x, \xi + i\psi).$$

This function is analytic where $f(\zeta)$ is analytic. To show this we first verify that the derivatives in the ζ and ψ directions are equal

$$\begin{aligned}\frac{\partial}{\partial \xi} g(\zeta) &= u_{xy}(x, \xi + i\psi) - iu_{yy}(x, \xi + i\psi) \\ -i\frac{\partial}{\partial \psi} g(\zeta) &= -i(iu_{xy}(x, \xi + i\psi) + u_{yy}(x, \xi + i\psi)) = u_{xy}(x, \xi + i\psi) - iu_{yy}(x, \xi + i\psi)\end{aligned}$$

Since these partial derivatives are equal and continuous, $g(\zeta)$ is analytic. We evaluate the function $g(\zeta)$ at $\zeta = -ix$. (Substitute $y = -ix$)

$$f'(2x) = u_x(x, -ix) - iu_y(x, -ix)$$

We make a change of variables to solve for $f'(x)$.

$$f'(x) = u_x\left(\frac{x}{2}, -i\frac{x}{2}\right) - iu_y\left(\frac{x}{2}, -i\frac{x}{2}\right)$$

If the expression is non-singular, then this defines the analytic function, $f'(z)$, on the real axis. The analytic continuation to the complex plane is

$$f'(z) = u_x\left(\frac{z}{2}, -i\frac{z}{2}\right) - iu_y\left(\frac{z}{2}, -i\frac{z}{2}\right)$$

Note that $\frac{d}{dz} 2u(z/2, -iz/2) = u_x(z/2, -iz/2) - iu_y(z/2, -iz/2)$.

We integrate the equation to obtain:

$$f(z) = 2u\left(\frac{z}{2}, -i\frac{z}{2}\right) + c$$

We know that the real part of an analytic function determines that function to within an additive constant. Assuming that the above expression is non-singular, we have found a formula for writing an analytic function in terms of its real part. With the same method, we can find how to write an analytic

function in terms of its imaginary part, v . We can also derive formulas if u and v are expressed in polar coordinates:

$$f(z) = u(r, \theta) + iv(r, \theta).$$

Result 2.4 If $f(z) = u(x, y) + iv(x, y)$ is analytic and the expressions are non-singular

$$f(z) = 2u\left(\frac{z}{2}, -i\frac{z}{2}\right) + \text{const}$$

$$f(z) = i2v\left(\frac{z}{2}, -i\frac{z}{2}\right) + \text{const.}$$

If $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic and the expressions are non-singular, then

$$f(z) = 2u\left(z^{1/2}, -\frac{i}{2}\log z\right) + \text{const}$$

$$f(z) = i2v\left(z^{1/2}, -\frac{i}{2}\log z\right) + \text{const.}$$

1

Example 2.4.11 Consider the problem of finding $f(z)$ given that

$$u(x, y) = e^{-x}(x \sin y - y \cos y).$$

$$\begin{aligned} f(z) &= 2u\left(\frac{z}{2}, -i\frac{z}{2}\right) \\ &= 2e^{-z/2}\left(\frac{z}{2}\sin\left(-i\frac{z}{2}\right) + i\frac{z}{2}\cos\left(-i\frac{z}{2}\right)\right) + c \\ &= iz e^{-z/2}\left(i\sin\left(i\frac{z}{2}\right) + \cos\left(-i\frac{z}{2}\right)\right) + c \\ &= iz e^{-z/2}(e^{-z/2}) + c \\ &= iz e^{-z} + c \end{aligned}$$

Example 2.4.12 Consider

$$\text{Log } z = \frac{1}{2}\text{Log}(x^2 + y^2) + i\text{Arctan}(x, y).$$

We try to construct the analytic function from its real part using the Equation

$$\begin{aligned}
f(z) &= 2u\left(\frac{z}{2}, -i\frac{z}{2}\right) + c \\
&= 2\frac{1}{2}\text{Log}\left(\left(\frac{z}{2}\right)^2 + \left(-i\frac{z}{2}\right)^2\right) + c \\
&= \text{Log}(0) + c
\end{aligned}$$

We obtain a singular expression, so the method fails.

Example 2.4.13: Again consider the logarithm, this time written in terms of polar coordinates (1)

$$\text{Log } z = \text{Log } r + i\theta$$

We try to construct the analytic function from its real part using Equation

$$\begin{aligned}
f(z) &= 2u\left(z^{1/2}, -i\frac{i}{2}\log z\right) + c \\
&= 2\text{Log}\left(z^{1/2}\right) + c \\
&= \text{Log } z + c
\end{aligned}$$

chapter 3

Analytic Continuation of Sums:

This chapter is about analytic continuation of summation ,analytic continuation along a curve and statement of Mondormy theorem which is an important result about analytic continuation of a complex analytic function to a larger set

3.1defenition:

Consider the function

$$f_1(z) = \sum_{n=0}^{\infty} z^n.$$

The sum converges uniformly for $D_1 = \{z \mid$

$|z| < r < 1\}$. Since the derivative also converges in this domain the function is analytic

There \square

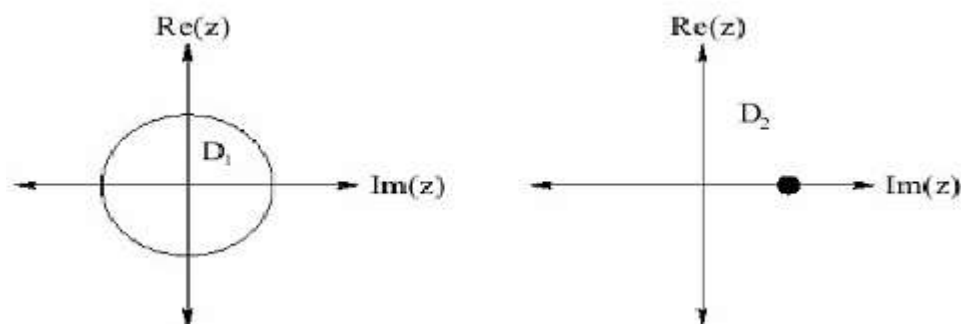


Fig 3.1

Now consider the function

$$f_2(z) = \frac{1}{1-z}.$$

This function is analytic everywhere except the point $z = 1$. On the domain $D_1 (1)$

$$f_2(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = f_1(z)$$

Analytic continuation tells us that there is a function that is analytic on the union of the two domains .here the domain is the entire z plane except the point $z = 1$ and the function is:

$$f(z) = \frac{1}{1-z}.$$

$\frac{1}{1-z}$ is said to be an analytic continuation of $\sum_{n=0}^{\infty} z^n$. [1]

3.2 Power series :

A series of geometrically increasing numbers

$$S_n = 1 + x + x^2 + x^3 + \dots + x^n$$

can be expressed in terms of just the second and the last number by noting that

$$\begin{aligned} 1 + xS_n &= 1 + x + x^2 + x^3 + \dots + x^n + x^{n+1} \\ &= S_n + x^{n+1} \end{aligned}$$

Solving for S_n gives

$$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + x^3 + \dots + x^n \quad (1)$$

Now, if the magnitude of x is less than 1, the quantity x^{n+1} goes to zero as n increases, so we immediately have the sum of the infinite geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad (2)$$

Archimedes evaluated the area enclosed by a parabola and a straight line essentially by determining the sum of such a series. This is perhaps the first example of a function being associated with the sum of an infinite number of terms. To illustrate, if we set x equal to $1/2$, this equation gives

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

There is, of course, a very significant difference between equations (1) and (2), because the former is valid for any value of x , whereas the latter clearly is not... at least not in the usual sense of finite arithmetical quantities. For example, if we set x equal to 2 in equation (2) we get

$$-1 = 1 + 2 + 4 + 8 + \dots$$

which is surely not a valid arithmetic equality in the usual sense, because the right side doesn't converge on any finite value (let alone $\square 1$). This shows that the correspondence between a function and an infinite series such as (2) may hold good only over a limited range of the variable. Generally speaking, an analytic function $f(z)$ can be expanded into a power series about any complex value of the variable z by means of Taylor's expansion, which can be written as

$$f(z_0 + z) = f(z_0) + \frac{f'(z_0)}{1!}z + \frac{f''(z_0)}{2!}z^2 + \frac{f'''(z_0)}{3!}z^3 + \dots$$

but the series will converge on the function only over a circular region of the complex plane centered on the point z_0 and extending to the nearest pole of the function (i.e., a point where the function goes to infinity). For example, the function $f(z) = 1/(1 - z)$ discussed previously has a pole at $z = 1$, so the disk of convergence of the power series for this function about the origin ($z = 0$) has a radius of 1. Hence the series given by (2) converges unconditionally only for values of x with magnitude less than 1. [4]

The analytic function $f(z) = 1/(1 - z)$ can also be expanded into a power series about any other point (where the function and its derivatives are well behaved). The derivatives of $f(z)$ are

$$f'(z) = \frac{1}{(1-z)^2} \quad f''(z) = \frac{2}{(1-z)^3} \quad f'''(z) = \frac{6}{(1-z)^4}$$

and so on. Inserting these into the expression for Taylor's series we get

$$f(z_0 + z) = \frac{1}{(1-z_0)} + \frac{z}{(1-z_0)^2} + \frac{z^2}{(1-z_0)^3} + \frac{z^3}{(1-z_0)^4} + \dots$$

Hence the power series for this function about the point $z_0 = 2$ is

$$f(2+z) = -1 + z - z^2 + z^3 - z^4 + \dots$$

Each of the power series obtained in this way is convergent only on the circular region of the complex plane centered on z_0 and extending to the nearest pole of the function. For example, since the function $f(z) = 1/(1-z)$ has a pole at $z = 1$, the power series with $z_0 = 2$ is convergent only in the shaded region shown in the figure below(3.2)

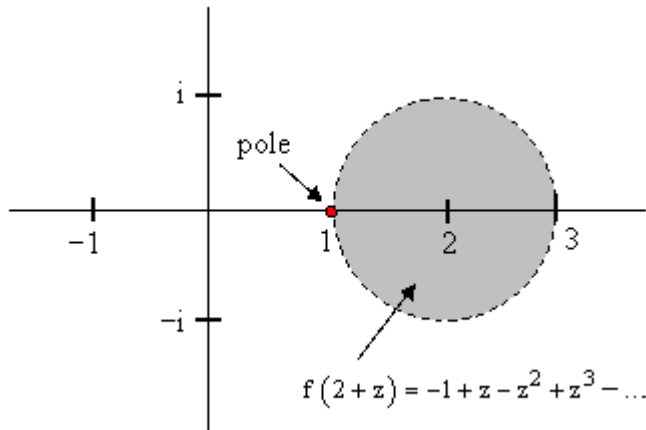


Fig 3.2

So far we've discussed only the particular function $f(z) = 1/(1-z)$ and we've simply shown how this known analytic function is equal to certain power series in certain regions of the complex plane. However, in some circumstances we may be given a power series having no explicit closed-form expression for the analytic function it represents (in its region of convergence). In such cases we can often still determine the values of the "underlying" analytic function for arguments outside the region of convergence of the given power series by a technique called analytic continuation.

To illustrate with a simple example, suppose we are given the power series.

$$f(z) = 1 + z + z^2 + z^3 + \dots$$

and suppose we don't know the closed-form expression for the analytic function represented by this series. As noted above, the series converges for values of z with magnitudes less than 1, but it diverges for values of z with magnitudes greater than 1. Nevertheless, by the process of analytic continuation we can determine the value of this function at any complex value of z . To do this, consider again the region of convergence for the given power series as shown below.

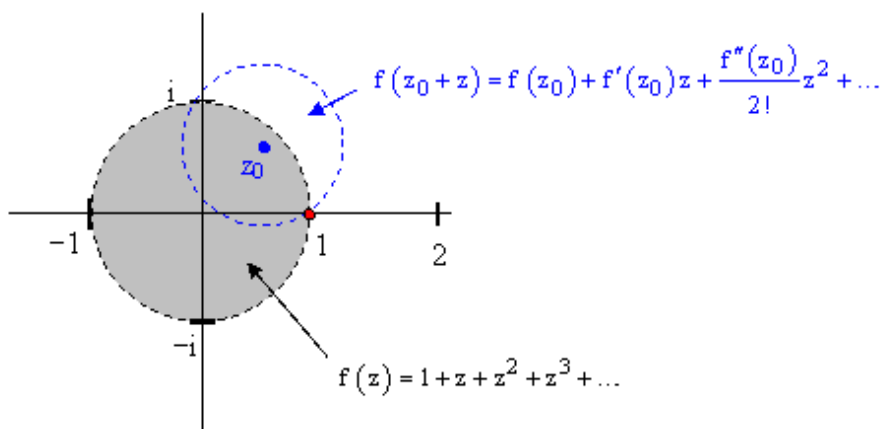


Fig 3.3

Since the known power series equals the function within its radius of convergence, we can evaluate $f(z)$ and its derivatives at any point in that region. Therefore, we can choose a point such as z_0 shown in the figure above(3.3), and determine the power series expression for $f(z_0 + z)$, which will be convergent within a circular region centered on z_0 and extending to the nearest pole. Thus we can now evaluate the function at values that lie outside the region of convergence of the original power series.

Once we have determine the power series for $f(z_0 + z)$ we can repeat the process by selecting a point z_1 inside the region of convergence and determining the power series for $f(z_1 + z)$, which will be convergent in a circular region centered on z_1 and extending to the nearest pole (which is at

$z = 1$ in this example). This is illustrated in the figure below. [4]

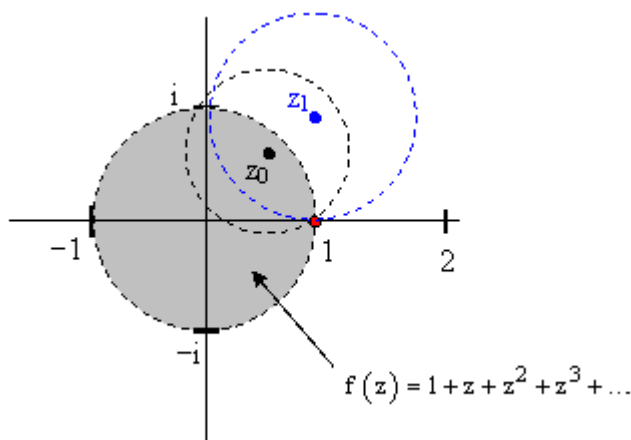


Fig 3.4

Continuing in this way, we can analytically extend the function throughout the entire complex plane, except where the function is singular, i.e., at the poles of the function. [4]

In general, given a power series of the form

$$f(z) = a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + a_3(z - \alpha)^3 + \dots$$

where the a_j are complex coefficients and α is a complex constant, we can express the same function as a power series centered on a nearby complex number β as

$$f(z) = b_0 + b_1(z - \beta) + b_2(z - \beta)^2 + b_3(z - \beta)^3 + \dots$$

where the b_j are complex coefficients. In order for these two power series to be equal for arbitrary values of z in this region, we must equate the coefficients of powers of z , so we must have

$$\begin{aligned}
a_0 - \alpha a_1 + \alpha^2 a_2 - \alpha^3 a_3 + \dots &= b_0 - \beta b_1 + \beta^2 b_2 - \beta^3 b_3 + \dots \\
a_1 - 2\alpha a_2 + 3\alpha^2 a_3 - 4\alpha^3 a_4 + \dots &= b_1 - 2\beta b_2 + 3\beta^2 b_3 - 4\beta^3 b_4 + \dots \\
a_2 - 3\alpha a_3 + 6\alpha^2 a_4 - 10\alpha^3 a_5 + \dots &= b_2 - 3\beta b_3 + 6\beta^2 b_4 - 10\beta^3 b_5 + \dots
\end{aligned}$$

In matrix form these conditions can be written as

$$\begin{bmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 & \alpha^4 & \dots \\ 0 & 1 & -2\alpha & 3\alpha^2 & -4\alpha^3 & \dots \\ 0 & 0 & 1 & -3\alpha & 6\alpha^2 & \dots \\ 0 & 0 & 0 & 1 & -4\alpha & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & -\beta & \beta^2 & -\beta^3 & \beta^4 & \dots \\ 0 & 1 & -2\beta & 3\beta^2 & -4\beta^3 & \dots \\ 0 & 0 & 1 & -3\beta & 6\beta^2 & \dots \\ 0 & 0 & 0 & 1 & -4\beta & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{bmatrix}$$

Multiplying through by the inverse of the right-hand coefficient matrix, this gives

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 & \dots \\ 0 & 1 & 2\varepsilon & 3\varepsilon^2 & 4\varepsilon^3 & \dots \\ 0 & 0 & 1 & 3\varepsilon & 6\varepsilon^2 & \dots \\ 0 & 0 & 0 & 1 & 4\varepsilon & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} \quad (3)$$

Where $\varepsilon = \beta - \alpha$. Naturally this is equivalent to applying Taylor's expansion. Now, it might seem as if this precludes any extension of the domain of the original power series. For example, suppose the original function was the power series for $1/(1 - z)$ about the point $\alpha = 0$, so the power series coefficients a_0, a_1, \dots would all equal 1. According to the above matrix equation the coefficient b_0 for the power series about the point β would be simply

$$b_0 = 1 + \beta + \beta^2 + \beta^3 + \dots$$

which of course converges only over the same region as the original power series. Also, it's of no help to split up the series transformation into smaller steps, because the compositions of the coefficient matrix are given by

$$\begin{bmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 & \dots \\ 0 & 1 & 2\varepsilon & 3\varepsilon^2 & 4\varepsilon^3 & \dots \\ 0 & 0 & 1 & 3\varepsilon & 6\varepsilon^2 & \dots \\ 0 & 0 & 0 & 1 & 4\varepsilon & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}^n = \begin{bmatrix} 1 & (n\varepsilon) & (n\varepsilon)^2 & (n\varepsilon)^3 & (n\varepsilon)^4 & \dots \\ 0 & 1 & 2(n\varepsilon) & 3(n\varepsilon)^2 & 4(n\varepsilon)^3 & \dots \\ 0 & 0 & 1 & 3(n\varepsilon) & 6(n\varepsilon)^2 & \dots \\ 0 & 0 & 0 & 1 & 4(n\varepsilon) & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Thus the net effect of splitting ε into n segments of size ε/n and applying the individual transformation n times is evidently identical to the effect of performing the transformation in a single step. From this we might conclude that it's impossible to analytically continue the power series $1 + z + z^2 + \dots$ to any point such as $3i/2$ with magnitude greater than 1. However, it actually is possible to analytically continue the geometric series, but only because of conditional convergence.

This is most easily explained with an example. Beginning with the power series

$$f(z) = 1 + z + z^2 + z^3 + \dots \tag{4}$$

centered on the origin, we can certainly express this as a power series centered on the complex number $\varepsilon = 3i/4$, because the power series $f(z)$ and

it derivatives are all convergent at this point (since it is inside the unit circle of convergence). By equation (3) with $a_0 = a_1 = a_2 = \dots = 1$, the coefficients of

$$f(z) = b_0 + b_1(z-\varepsilon) + b_2(z-\varepsilon)^2 + b_3(z-\varepsilon)^3 + \dots$$

Are

$$b_0 = 1 + \left(\frac{3i}{4}\right) + \left(\frac{3i}{4}\right)^2 + \left(\frac{3i}{4}\right)^3 + \dots = \frac{16+12i}{25}$$

$$b_1 = 1 + 2\left(\frac{3i}{4}\right) + 3\left(\frac{3i}{4}\right)^2 + 4\left(\frac{3i}{4}\right)^3 + \dots = \frac{112+384i}{625}$$

$$b_2 = 1 + 3\left(\frac{3i}{4}\right) + 6\left(\frac{3i}{4}\right)^2 + 10\left(\frac{3i}{4}\right)^3 + \dots = \frac{-2816+7488i}{15625}$$

etc.

The absolute values of these coefficients are $b_n = (4/5)^{n+1}$. Now if we take these as the a_n values and apply the same transformation again, shifting the center of the power series by another $\varepsilon = 3i/4$, so that the resulting series is centered on $3i/2$, we find that the zeroth coefficient given by equation (3) is

$$\begin{aligned} b_0 &= \left[\frac{16+12i}{25}\right] + \left(\frac{3i}{4}\right)\left[\frac{112+384i}{625}\right] + \left(\frac{3i}{4}\right)^2\left[\frac{-2816+7488i}{15625}\right] + \dots \\ &= \frac{4+6i}{13} \end{aligned}$$

in agreement with the analytic expression for the function. This series clearly converges, because each term has geometrically decreasing magnitude. Similarly we can compute the higher order coefficients for the power series centered on the point $3i/2$, which is well outside the radius of

convergence of the original geometric series centered on the origin. But how can this be? We've essentially just multiplied the unit column vector by the coefficient vector for ε twice, which we now gives the divergent result [4]

$$b_0 = 1 + \left(\frac{3}{2}i\right) + \left(\frac{3}{2}i\right)^2 + \left(\frac{3}{2}i\right)^3 + \left(\frac{3}{2}i\right)^4 + \dots$$

To examine this more closely, let us expand the quantities in the square brackets in the preceding expression for b_0 . This gives

$$\begin{aligned} b_0 = & 1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \dots \\ & + \left(\frac{3}{4}\right) + 2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right)^3 + 4\left(\frac{3}{4}\right)^4 + \dots \\ & + \left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right)^3 + 6\left(\frac{3}{4}\right)^4 + 10\left(\frac{3}{4}\right)^5 + \dots \\ & + \left(\frac{3}{4}\right)^3 + 4\left(\frac{3}{4}\right)^4 + 10\left(\frac{3}{4}\right)^5 + 20\left(\frac{3}{4}\right)^6 + \dots \\ & \text{etc.} \end{aligned}$$

Each individual row is convergent, and moreover the rows converge on geometrically decreasing values, so the sum of the sums of the rows is also convergent. However, if we sum the individual values by diagonals we get

$$\begin{aligned} b_0 = & 1 + \left[\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\right] + \left[\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right] \\ & + \left[\left(\frac{3}{4}\right)^3 + 3\left(\frac{3}{4}\right)^3 + 3\left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^3\right] + \dots \\ = & 1 + \left(\frac{3}{2}i\right) + \left(\frac{3}{2}i\right)^2 + \left(\frac{3}{2}i\right)^3 + \left(\frac{3}{2}i\right)^4 + \dots \end{aligned}$$

Thus the terms for b_0 are divergent if we sum them diagonally, but they are convergent if we sum them by rows. In other words, the series is conditionally convergent, which is to say, the sum of the series – and even whether it sums to a finite value at all – depends on the order in which we sum the terms. The same applies to the series for the other coefficients.

Since the terms of a conditionally convergent series can be re-arranged to give any value we choose, one might wonder if analytic continuation – which is based so fundamentally on conditional convergence – really gives a unique result. The answer is yes, but only because we carefully stipulate the procedure for transforming the power series coefficients in such a way as to cause the sums to be evaluated “by rows” and not in any other way. This procedure is justified mainly by the fact that it gives results that agree with the explicit analytic functions in cases when those functions are known.

To show that we can also continue the geometric series to points on the other side of the singularity using this procedure, consider again the initial power series (4), and this time suppose we determine the sequence of series centered on points located along the unit circle centered on the point $z = 1$ as indicated in the figure 3.5 . [4]

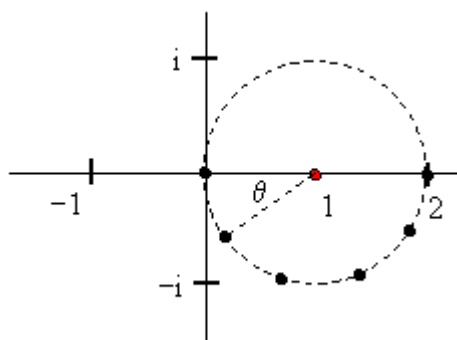


Fig 3.5

Thus, letting $\alpha = e^{i\theta}$ we wish to carry out successive shifts of the power series center by the increments

$$\varepsilon_1 = 1 - \alpha \quad \varepsilon_2 = \alpha(1 - \alpha) \quad \varepsilon_3 = \alpha^2(1 - \alpha) \quad \varepsilon_4 = \alpha^3(1 - \alpha)$$

and so on. Applying equation (3) with $\varepsilon = \varepsilon_1$ to perform the first of these transformations we get the sequence of coefficients (4)

$$\begin{aligned} b_0 &= 1 + (1 - \alpha) + (1 - \alpha)^2 + (1 - \alpha)^3 + \dots = \frac{1}{\alpha} \\ b_1 &= 1 + 2(1 - \alpha) + 3(1 - \alpha)^2 + 4(1 - \alpha)^3 + \dots = \frac{1}{\alpha^2} \\ b_2 &= 1 + 3(1 - \alpha) + 6(1 - \alpha)^2 + 10(1 - \alpha)^3 + \dots = \frac{1}{\alpha^3} \\ &\vdots \end{aligned}$$

Now if we call these the α_j coefficients, and perform the next transformation using equation (3) with $\varepsilon = \varepsilon_2$, we get

$$\begin{aligned} b_0 &= \frac{1}{\alpha} + [\alpha(1 - \alpha)] \frac{1}{\alpha^2} + [\alpha(1 - \alpha)]^2 \frac{1}{\alpha^3} + [\alpha(1 - \alpha)]^3 \frac{1}{\alpha^4} + \dots = \frac{1}{\alpha^2} \\ b_1 &= \frac{1}{\alpha^2} + 2[\alpha(1 - \alpha)] \frac{1}{\alpha^3} + 3[\alpha(1 - \alpha)]^2 \frac{1}{\alpha^4} + 4[\alpha(1 - \alpha)]^3 \frac{1}{\alpha^5} + \dots = \frac{1}{\alpha^4} \\ b_2 &= \frac{1}{\alpha^3} + 3[\alpha(1 - \alpha)] \frac{1}{\alpha^4} + 6[\alpha(1 - \alpha)]^2 \frac{1}{\alpha^5} + 10[\alpha(1 - \alpha)]^3 \frac{1}{\alpha^6} + \dots = \frac{1}{\alpha^6} \\ &\vdots \end{aligned}$$

Each of these sums is clearly convergent, because $|\alpha| = 1$ and $|1 - \alpha| < 1$.

Continuing in this way, the n th power series in this sequence is

$$f_n(z) = e^{-ni\theta} + e^{-2ni\theta}(z - \mu_n) + e^{-3ni\theta}(z - \mu_n)^2 + e^{-4ni\theta}(z - \mu_n)^3 + \dots$$

where $\mu_n = (1 - e^{ni\theta})$ is the n th point around the circle. This can also be written in the form

$$\begin{aligned} f_n(z) &= e^{-ni\theta} \left(1 + \left(\frac{z - \mu_n}{e^{ni\theta}} \right) + \left(\frac{z - \mu_n}{e^{ni\theta}} \right)^2 + \left(\frac{z - \mu_n}{e^{ni\theta}} \right)^3 + \dots \right) \\ &= \frac{e^{-ni\theta}}{1 - \frac{z - \mu_n}{e^{ni\theta}}} = \frac{1}{1 - z} \end{aligned}$$

These examples demonstrate that equation (3) can be used consistently to give the analytic continuations of power series, although in cases where the sums cannot be explicitly identified by closed-form expressions there is a problem of sensitivity to the precision of the initial conditions and the subsequent computations. At each stage we need to evaluate infinite series, and the higher order coefficients tend to require more and more terms before they converge, and there are infinitely many coefficients to evaluate. If we limit our calculations to (say) just the first 1000 coefficients, the effect of the unspecified coefficients will propagate to c_0 in about 1000 steps. Smaller incremental steps require fewer terms for convergence of each sum, but they also require more transformations to reach any given point, and this necessitates carrying a larger number of coefficients. So, in practice, the pure numerical transformation of series using equation (3) often leads to difficulties. It's also worth noting that many power series possess a "natural boundary", i.e., the region of convergence is enclosed by a continuous locus of points at which the function is singular or not well-behaved in some other

sense (e.g., not differentiable), and this prevents analytic continuation of the series. Nevertheless, it's interesting that an analytic function can, at least formally, be represented by a field of infinite-dimensional complex vectors, and that the process of analytic continuation can be represented by non-associative matrix multiplication. The failure of associativity is due to the fact that the convergence of the conditionally convergent series depends on the order in which we add the terms, and this depends on the order in which the matrix multiplications are performed.[4]

Incidentally, in each when analytically continuing the geometric series $f(z) = 1 + z + z^2 + z^3 + \dots$ by the procedure described above, we could have noted that the transformed functions centered on the point z_0 are expressible in the form

$$f(z) = f(z_0) f\left(\frac{z-z_0}{1-z_0}\right)$$

This is a simple functional equation, and it can be applied recursively to give the analytic continuation of the function to all points on the complex plane (except for the pole at $z = 1$). For any z we can choose a value of z_0 that is close enough to z so that the absolute value of $(z - z_0)/(1 - z_0)$ is less than 1 and hence the function f of that value converges. Of course, to apply the above equation we must also be able to evaluate $f(z_0)$, even if the magnitude of z_0 exceeds 1, but we can do this by applying the formula again. For example, if we wish to evaluate $f(3i)$ we could use the power series centered on $z_1 = 7i/4$, which requires us to evaluate $f(7i/4)$, and this can be done using the power series centered on $z_0 = 3i/4$. Thus we can write

$$f(z) = f\left(\frac{z_0 - 0}{1 - 0}\right) f\left(\frac{z_1 - z_0}{1 - z_0}\right) f\left(\frac{z - z_1}{1 - z_1}\right)$$

The argument of each of the right-hand functions has magnitude less than 1, so they can each be evaluated using the original geometric series to give $f(3i) = 0.1 + 0.3i$, which naturally agrees with the value $1/(1-3i)$. In general, to evaluate $f(z_n)$ for any arbitrary value of z_n , we could split up a path from the origin to z_n into n small increments Δz and then multiply together the values of $f(\Delta z/(1-z))$ to give the overall result. If we take the natural log of both sides, the expression could be written in the form

$$\ln[f(z_n)] = \ln\left[f\left(\frac{\Delta z_0}{1-z_0}\right)\right] + \ln\left[f\left(\frac{\Delta z_1}{1-z_1}\right)\right] + \dots + \ln\left[f\left(\frac{\Delta z_{n-1}}{1-z_{n-1}}\right)\right]$$

In the limit as the increments become arbitrarily small we can replace Δz with dz and integrate the right hand side. In this limiting case only the first-order term of the geometric series is significant, so we have

$$f\left(\frac{dz}{1-z}\right) \rightarrow 1 + \frac{dz}{1-z} \qquad \ln\left[1 + \frac{dz}{1-z}\right] \rightarrow \frac{dz}{1-z}$$

Therefore the integral of the right hand side reduces to

$$\ln[f(z_n)] = \int_0^{z_n} \frac{dz}{1-z} = \ln\left(\frac{1}{1-z_n}\right)$$

from which it follows that

$$f(z_n) = \frac{1}{1-z_n}$$

Another important aspect of analytic continuation is the fact that the continuation of a given power series to some point outside the original region of convergence can lead to different values depending on the path taken. This phenomenon didn't arise in our previous examples, because the analytic function $1/(1-z)$ is single-valued over the entire complex plain, but some functions are found to be multi-valued when analytically continued. To illustrate, consider the power series [4]

$$f(z) = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \dots$$

which of course equals $\ln(z)$ within the region of convergence. This series is centered about the point $z = 1$, and at $z = 0$ it yields the negative of the harmonic series, which diverges, so the function is singular at $z = 0$. Now suppose we analytically continue this power series to a sequence of power series centered on points on the unit circle around the origin, i.e., the sequence of points $e^{i\theta}, e^{2i\theta}, e^{3i\theta}, \dots$ for some constant angle θ . Noting that the n th derivative of $\ln(z)$ is

$$\frac{d^n}{dz^n} \ln(z) = (-1)^{n-1} \frac{(n-1)!}{z^n}$$

we see that the power series centered on the point $e^{i\theta}$ is given by the Taylor series expansion

$$\begin{aligned}
f_1(z) &= f(e^{i\theta}) + \frac{1}{1!} f'(e^{i\theta})(z - e^{i\theta}) + \frac{1}{2!} f''(e^{i\theta})(z - e^{i\theta})^2 + \frac{1}{3!} f'''(e^{i\theta})(z - e^{i\theta})^3 + \dots \\
&= i\theta + e^{-i\theta}(z - e^{i\theta}) - \frac{e^{-2i\theta}}{2}(z - e^{i\theta})^2 + \frac{e^{-3i\theta}}{3}(z - e^{i\theta})^3 - \dots \\
&= i\theta + \left(\frac{z}{e^{i\theta}} - 1\right) - \frac{1}{2}\left(\frac{z}{e^{i\theta}} - 1\right)^2 + \frac{1}{3}\left(\frac{z}{e^{i\theta}} - 1\right)^3 - \dots \\
&= i\theta + f\left(\frac{z}{e^{i\theta}}\right)
\end{aligned}$$

Repeating this calculation for each successive point, we find

$$f_n(z) = ni\theta + f\left(\frac{z}{e^{ni\theta}}\right)$$

This converges provided $|z - e^{ni\theta}| < 1$. For $n\theta = 2\pi k$ we have $f_n(z) = (2\pi i)k + f(z)$, so each time we circle the singularity at the origin the value of the function increases by $2\pi i$. This is consistent with the fact that the natural log function (i.e., the inverse of the exponential function) of any given complex number has infinitely many values, separated by $2\pi i$.

Notice that, in this case, “functional equation” is simply

$$f(z) = f(z_0) + f\left(\frac{z}{z_0}\right)$$

which can be used in a way analogous to how the functional equation for the geometric series was used to analytically continue the power series to all non-singular points. In this regard, it’s interesting to recall that the matrix formulation given by equation (3) is entirely generic, and applies to all

power series, represented as infinite dimensional vectors, so whether or not a certain power series continues to a single-valued function (like the geometric series), a multi-valued function (like the series for the natural log), or can't be continued at all, depends entirely on the initial "vector". [4]

3.2 Analytic continuation along a curve

The definition of analytic continuation along a curve is a bit technical, but the basic idea is that one starts with an analytic function defined around a point, and one extends that function along a curve via analytic functions defined on small overlapping disks covering that curve.

Formally, consider a curve (a continuous function) $\gamma: [0, 1] \rightarrow \mathbb{C}$. Let f be an analytic function defined on an open disk U centered at $\gamma(0)$. An analytic continuation of the pair (f, U) along γ is a collection of pairs (f_t, U_t) for $0 \leq t \leq 1$ such that

$$F_0 = f \text{ and } U_0 = U$$

For each $t \in [0, 1]$, U_t is an open disk centered at $\gamma(t)$ and $f_t: U_t \rightarrow \mathbb{C}$ is an analytic function

For each $t \in [0, 1]$ there exists $\varepsilon > 0$ such that for all $t' \in [0, 1]$ with $|t - t'| < \varepsilon$ one has that $\gamma(t') \in U_t$ (which implies that U_t and $U_{t'}$ have a non-empty intersection) and the functions f_t and $f_{t'}$ coincide on the intersection $U_t \cap U_{t'}$. [5]

3.3 Properties of analytic continuation along a curve

Analytic continuation along a curve is essentially unique, in the sense that given two analytic continuations (f_t, U_t) and (g_t, V_t) ($0 \leq t \leq 1$) of (f, U) along

γ , the functions f_1 and g_1 coincide on $U_1 \cap V_1$. Informally, this says that any two analytic continuations of (f,U) along γ will end up with the same values in a neighborhood of $\gamma(1)$.

If the curve γ is closed (that is, $\gamma(0) = \gamma(1)$), one need not have f_0 equal f_1 in a neighborhood of $\gamma(0)$. For example, if one starts at a point $(a,0)$ with $a > 0$ and the complex logarithm defined in a neighborhood of this point, and one lets γ be the circle of radius a centered at the origin (traveled counterclockwise from $(a,0)$), then by doing an analytic continuation along this curve one will end up with a value of the logarithm at $(a,0)$ which is $2\pi i$ plus the original value. [5]

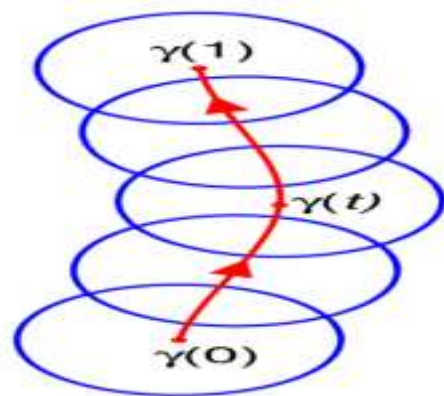


fig 3.6

3.4 Monodromy theorem

In mathematics, more precisely in complex analysis, the monodromy theorem is an important result about analytic continuation of a complex-analytic function to a larger set. The idea is that one can extend a complex-analytic function along curves starting in the original domain of the function and ending in the larger set. A potential problem of this analytic continuation along a curve strategy is there are usually many curves which end up at the same point in the larger set. The monodromy theorem gives sufficient conditions for analytic continuation to give the same value at a given point

regardless of the curve used to get there, so that the resulting extended analytic function is well-defined and single-valued.

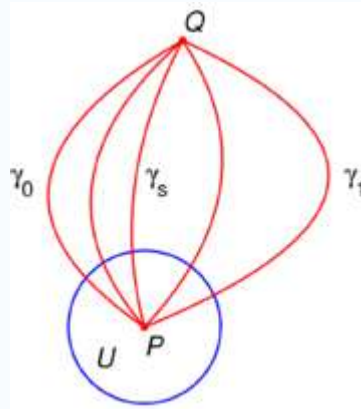


Fig 3.7

As noticed earlier, two analytic continuations along the same curve yield the same result at the curve's endpoint. However, given two different curves branching out from the same point around which an analytic function is defined, with the curves reconnecting at the end, it is not true in general that the analytic continuations of that function along the two curves will yield the same value at their common endpoint.

Indeed, one can consider, as in the previous section, the complex logarithm defined in a neighborhood of a point $(a,0)$ and the circle centered at the origin and radius a . Then, it is possible to travel from $(a,0)$ to $(-a,0)$ in two ways, counterclockwise, on the upper half-plane arc of this circle, and clockwise, on the lower half-plane arc. The values of the logarithm at $(-a,0)$ obtained by analytic continuation along these two arcs will differ by $2\pi i$.

If, however, one can continuously deform one of the curves into another while keeping the starting points and ending points fixed, and analytic continuation is possible on each of the intermediate curves, then the analytic continuations along the two curves will yield the same results at their

common endpoint. This is called the monodromy theorem and its statement is precise made below

Let U be an open disk in the complex plane centered at a point P and $f : U \rightarrow \mathbb{C}$ be a complex-analytic function. Let Q be another point in the complex plane. If there exists a family of curves $\gamma_s : [0, 1] \rightarrow \mathbb{C}$ with $s \in [0, 1]$ such that $\gamma_s(0) = P$ and $\gamma_s(1) = Q$ for all $s \in [0, 1]$, the function $(s, t) \in [0, 1] \times [0, 1] \rightarrow \gamma_s(t) \in \mathbb{C}$ is continuous, and for each $s \in [0, 1]$ it is possible to do an analytic continuation of f along γ_s , then the analytic continuations of f along γ_0 and γ_1 will yield the same values at Q .

The monodromy theorem makes it possible to extend an analytic function to a larger set via curves connecting a point in the original domain of the function to points in the larger set. The theorem below which states that is also called the monodromy theorem.

Let U be an open disk in the complex plane centered at a point P and $f : U \rightarrow \mathbb{C}$ be a complex-analytic function. If W is an open simply-connected set containing U , and it is possible to perform an analytic continuation of f on any curve contained in W which starts at P , then f admits a direct analytic continuation to W , meaning that there exists a complex-analytic function $g : W \rightarrow \mathbb{C}$ whose restriction to U is f . [5]

Example :

In the process of analytic continuation, a function that is an analytic function $F(z)$ in some open subset E of the disk D given by

$$0 < |z| < 1$$

may be continued back into E , but with different values. For example if we take

$$F(z) = \log z$$

and E to be defined by

$$\operatorname{Re}(z) > 0$$

then analytic continuation anti-clockwise round the circle

$$|z| = 0.5$$

will result in the return, not to $F(z)$ but

$$F(z) + 2\pi i.$$

In this case the Monodromy group is infinite and the covering space is the universal cover of the punctured complex plane. This cover can be visualized to $\rho > 0$. The covering map is a vertical projection, in a sense collapsing the spiral in the obvious way to get a punctured plane.

chapter 4

Applications

Complex analysis has an important applications in modern physics, and a great variety of problems in physics--both conceptual and technical--can be explored by using it.also in biology.

4.1 Applications in complex functions :

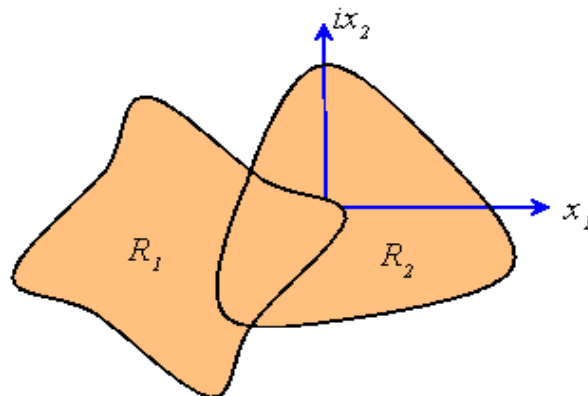
Acommon way to define functions in complex analysis proceeds by first specifying the function on a small domain only, and then extending it by analytic continuation. In practice, this continuation is often done by first establishing some functional equation on the small domain and then using this equation to extend the domain. Examples are the Riemann zeta function and the gamma function.

The concept of a universal cover was first developed to define a natural domain for the analytic continuation of an analytic function. The idea of finding the maximal analytic continuation of a function in turn led to the development of the idea of Riemann surfaces [3]

4-2 Solving Half-Plane Problems Using Analytic Continuation:

Some of the most interesting boundary value problems in linear elasticity have been solved using the idea of analytic continuation, which reduces many boundary value problems to a so-called Hilbert problem, with a known solution. Examples include the displacement and traction boundary value problem for the half-plane and the disk; contact problems (both for half-spaces and disks); crack problems (including cracks on the interfaces between dissimilar solids); and problems involving dislocations interacting with boundaries..

The Continuation Theorem



4-1

Suppose that $f_1(z)$ and $f_2(z)$ are analytic functions in regions R_1 and R_2 . Suppose that the two regions intersect in a domain R and there exists an infinite sequence z_n of discrete points in R with at least one limit point on which

$$f_1(z_n) = f_2(z_n) \quad n = 1, 2, \dots$$

Then the function

$$f(z) = \begin{cases} f_1(z) & z \in R_1 \\ f_2(z) & z \in R_2 \end{cases}$$

is analytic in the union of R_1 and R_2 . The function is $f_2(z)$ said to be the analytic continuation $f_1(z)$ of into R_2 ; similarly, $f_1(z)$ is the analytic continuation of $f_2(z)$ in R_1 .

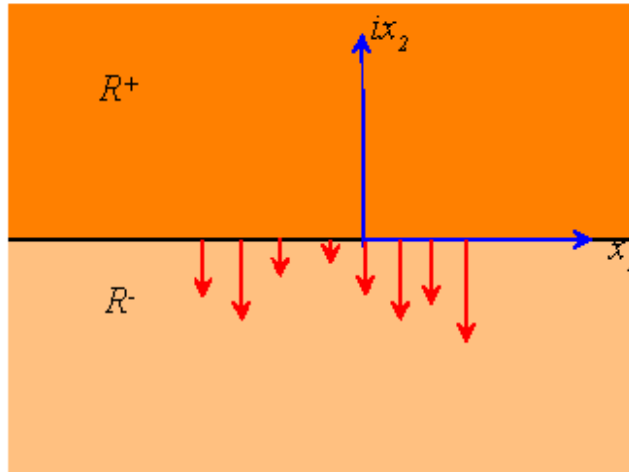
For our purposes, we will be considering regions that intersect along a line L , and $f_1(z) = f_2(z)$ along L . In this case $f(z)$ is analytic in $R_1 + R_2 + L$

The idea of analytic continuation provides a powerful tool for solving half-plane problems, and can also be used to solve problems involving regions with circular boundaries. We will illustrate the technique by using it to solve half-plane problems here.

The method of stress continuation for a half-plane

For this problem, we will attempt to determine the fields inside a half-space subjected to a prescribed distribution of traction on its surface, as shown in the picture.

Suppose that the region of interest is the upper half-plane, which we will refer to as R_+ .



4-2

The problem will be solved using analytic continuation. [5]

Here is the basic idea. In the usual formulation, we need to find two complex potentials, Ω

and ω . However, when we solve a half-plane problem, the potentials in R^- (the lower half-plane) are arbitrary – we can choose the potentials in R^- in any way we like, without changing the stress and displacement fields in R^+ . This observation allows us to find an analytic continuation of Ω in R^- (i.e. we find a function that is analytic in both R^+ and R^-), and then using the definition of Ω in R^- to replace ω in R^+ . Then, instead of having to find two analytic functions in R^+ , we need to find one potential Ω that is analytic in both R^+ and R^- , and satisfies certain boundary conditions on the real line.

For example, we will show that the following representation will generate displacement and stress fields in R^+ with the surface of the half-space free of tractions. Let Ω be analytic in both R^+ and R^- , and set

$$2\mu D = (3 - 4\nu)\Omega(z) + \Omega(\bar{z}) + (\bar{z} - z) \overline{\Omega'(z)}$$

$$\begin{aligned}\sigma_{11} + \sigma_{22} &= 2(\Omega'(z) + \overline{\Omega'(z)}) \\ \sigma_{22} - i\sigma_{12} &= \Omega'(z) - \overline{\Omega'(z)} + (z - \bar{z})\overline{\Omega''(z)}\end{aligned}$$

You can check that the surface is free from traction by letting the imaginary part of z approach zero (from above) in the expressions listed above.

The precise way we define Ω in R^- is arbitrary. We usually look for a definition that will make the resulting boundary conditions as simple as possible.

There is a systematic approach you can follow to devise an appropriate continuation, however, which we will illustrate for the case of a traction boundary value problem. [5]

Let Ω, ω be analytic in R^+ , and set

$$\begin{aligned}2\mu D &= (3 - 4\nu)\Omega(z) - z \overline{\Omega'(z)} - \overline{\omega(z)} \\ \sigma_{11} + \sigma_{22} &= 2(\Omega'(z) + \overline{\Omega'(z)}) \\ \sigma_{11} - \sigma_{22} + 2i\sigma_{12} &= -2(z \overline{\Omega''(z)} + \overline{\omega'(z)})\end{aligned}$$

Now, suppose that some region of the real axis is unstressed, i.e.

$$\sigma_{22} - i\sigma_{12} = 0 \quad \text{on } L$$

Where L is a region on the surface.

In terms of the complex potentials

$$\lim_{x_2 \rightarrow 0^+} \{\Omega'(z) + \overline{\Omega'(z)} + z \overline{\Omega''(z)} + \overline{\omega'(z)}\} = 0$$

We will introduce the notation

$$\Omega_+(x_1) \equiv \lim_{x_2 \rightarrow 0^+} \Omega(z)$$

whence

$$\Omega'_+(x_1) + \overline{\Omega'_+(x_1)} + x_1 \overline{\Omega''_+(x_1)} + \overline{\omega'_+(x_1)} = 0$$

This boundary condition may be re-written in terms of the functions

$$\overline{\Omega(\bar{z})}, \quad \overline{\omega(\bar{z})}$$

which are analytic in R^- .

$$\overline{\Omega}(z) \equiv \overline{\Omega(\bar{z})}, \quad \overline{\omega}(z) \equiv \overline{\omega(\bar{z})}$$

but in my view this is confusing and we won't use it here.

$$\lim_{x_2 \rightarrow 0^-} \left\{ \overline{\Omega'(\bar{z})} + \Omega'(\bar{z}) + \bar{z} \overline{\Omega''(\bar{z})} + \overline{\omega'(\bar{z})} \right\} = 0$$

Now, observe that

$$\lim_{x_2 \rightarrow 0^-} \overline{\Omega(\bar{z})} = \lim_{x_2 \rightarrow 0^+} \overline{\Omega(z)} = \overline{\Omega_+(z)}$$

Whence, substituting back into the preceding equation and taking complex conjugates:

$$\Omega'_+(z) = - \lim_{x_2 \rightarrow 0^-} \left\{ \overline{\Omega'(\bar{z})} + z \overline{\Omega''(\bar{z})} + \overline{\omega'(\bar{z})} \right\} \quad \text{on } x_2 = 0$$

This condition is equivalent to the statement that the function Ω' (which is analytic in R^+) has the same value as the function $\overline{\Omega'(\bar{z})} + z \overline{\Omega''(\bar{z})} + \overline{\omega'(\bar{z})}$ (which is analytic in R^-) on a line segment L . Therefore, by the continuation

theorem, these two functions continue one another analytically across the line L. We can therefore think of these two functions as a single complex potential, which is analytic everywhere, and set [5]

$$\Omega'(z) = \begin{cases} \Omega'(z) & z \in R^+ \\ -\overline{\Omega'(\bar{z})} - z\overline{\Omega''(\bar{z})} - \overline{\omega'(\bar{z})} & z \in R^- \end{cases}$$

We can integrate these equations to see that

$$\Omega(z) = \begin{cases} \Omega(z) & z \in R^+ \\ -z\overline{\Omega'(\bar{z})} - \overline{\omega(z)} & z \in R^- \end{cases}$$

We can use this result to find an expression for ω in R^+ :

$$\omega(z) = -\overline{\Omega(\bar{z})} - z\overline{\Omega'(z)} \quad z \in R^+$$

Finally, eliminate ω from our expression for displacements and stresses in R^+ to obtain

$$\begin{aligned} 2\mu D &= (3-4\nu)\Omega(z) + \Omega(\bar{z}) + (\bar{z}-z)\overline{\Omega'(z)} \\ \sigma_{11} + \sigma_{22} &= 2(\Omega'(z) + \overline{\Omega'(\bar{z})}) \\ \sigma_{22} - i\sigma_{12} &= \Omega'(z) - \overline{\Omega'(\bar{z})} + (z-\bar{z})\overline{\Omega''(\bar{z})} \end{aligned}$$

4.2 ·1 Traction Boundary Value Problems for the Half-Plane

We will Find the displacements and stresses in a half-space due to a prescribed distribution of traction on its surface.

We will use the representation based on stress continuation to derive our result. Evidently, we need to find a potential $\Omega(z)$ which is analytic in both R^+ and R^- , and satisfies

$$\begin{aligned} \sigma_{22} - i\sigma_{12} &= -(t_2^*(x_1) - it_1^*(x_1)) = iT^*(x_1) \\ \Rightarrow \lim_{x_2 \rightarrow 0^+} \left\{ \Omega'(z) - \overline{\Omega'(\bar{z})} + (z - \bar{z})\overline{\Omega''(z)} \right\} &= \Omega'_+(x_1) - \Omega'_-(x_1) = iT^*(x_1) \end{aligned}$$

(we assumed that

$$\lim_{x_2 \rightarrow 0^+} (z - \bar{z})\overline{\Omega''(z)} = 0$$

Thus, we need to find a potential with a prescribed discontinuity on the real line.

It looks like we are stumped here, but actually this problem has a well-known solution.

To solve the problem, we make use of the Plemelj formulae, for which we will need some more results from the general theory

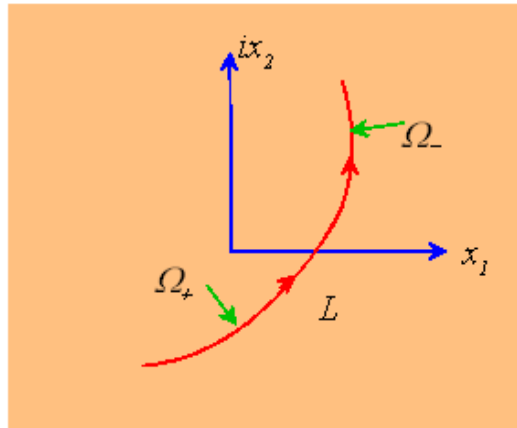
of complex variables [5]

The Plemelj formulae

Let $\phi(\zeta)$ be a complex valued function defined on an arc L . Assume that $\phi(\zeta)$ satisfies the Holder conditions on L , that is:

$$|\phi(\zeta_1) - \phi(\zeta_2)| < A|\zeta_1 - \zeta_2|^\kappa$$

for any two points $(\zeta_1, \zeta_2) \in L$, where A is a positive constant and $0 < \kappa < 1$



4-3

Then

$$\Omega(z) = \frac{1}{2\pi i} \int \frac{\phi(\zeta)}{\zeta - z} d\zeta$$

is sectionally analytic in a region R cut along L; that is to say, $\Omega(z)$ satisfies the following conditions:

(1) $\Omega(z)$ is analytic on $\{R-L\}$

(2) $\Omega(z)$ is sectionally continuous in the neighborhood of L

(3) At an end z_0 of the arc L, $\Omega(z)$ satisfies $\lim_{z \rightarrow z_0} (z - z_0)\Omega(z) = 0$

Furthermore, the limiting values Ω_{\pm} may be shown to exist on L and satisfy

$$\Omega_+(z_0) - \Omega_-(z_0) = \phi(z_0)$$

$$\Omega_+(z_0) + \Omega_-(z_0) = \frac{1}{\pi i} PV \int_L \frac{\phi(z)}{z - z_0} dz$$

Where PV denotes that the integral should be interpreted as a Cauchy Principal Value (the integral is singular because z_0 lies on L). These two equations are the Plemelj formula. These results provide the key to solving the half-plane problem. The most general solution satisfying our boundary condition is

$$\Omega'(z) = \frac{1}{2\pi} \int_L \frac{T^*(\zeta)}{\zeta - z} d\zeta + \sum_{n=0}^{\infty} \alpha_n z^n$$

We added the polynomial here because any continuous analytic function on R evidently generates zero traction on the surface. If the stresses vanish at infinity then $\alpha_n = 0$. [5]

Example: Determine the potentials generating displacement and stress fields in a half - plane loaded by uniform pressure p and shear s on the region $-a < x_1 < a$.

Evidently

$$\Omega'(z) = \frac{1}{2\pi} \int_{-a}^a \frac{s + ip}{\zeta - z} d\zeta = \frac{s + ip}{2\pi} \log\left(\frac{z - a}{z + a}\right)$$

And hence

$$\Omega(z) = \frac{s + ip}{2\pi} \{(z - a)\log(z - a) - (z + a)\log(z + a)\}$$

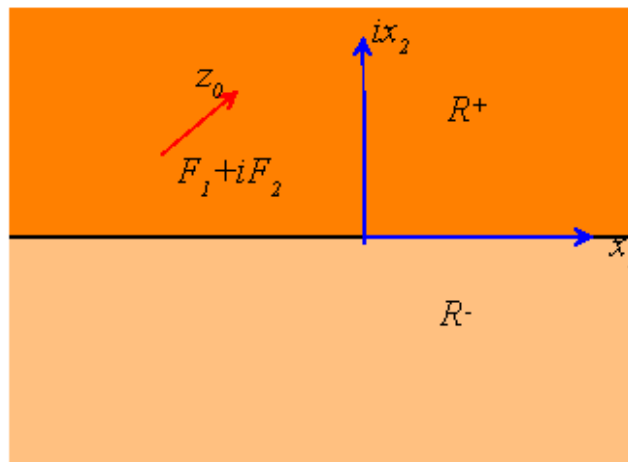
Displacement and stress fields may be then determined from

$$2\mu D = (3 - 4\nu)\Omega(z) + \Omega(\bar{z}) + (\bar{z} - z) \overline{\Omega'(z)}$$

$$\sigma_{11} + \sigma_{22} = 2(\Omega'(z) + \overline{\Omega'(\bar{z})})$$

$$\sigma_{22} - i\sigma_{12} = \Omega'(z) - \Omega'(\bar{z}) + (z - \bar{z})\overline{\Omega''(z)}$$

Example: Fields induced by point forces beneath the surface of a half-plane. Here is another useful trick which exploits the idea of analytic continuation.



4-4

A half-plane with traction free surface is subjected to a point force $F_1 + iF_2$ acting at a point z_0 . Determine complex potentials for this problem.

Recall that the potentials

$$\Omega_0(z) = -\frac{F_1 + iF_2}{8\pi(1-\nu)} \log(z - z_0) \quad \omega_0(z) = -\frac{(3-4\nu)(F_1 + iF_2)}{8\pi(1-\nu)} \log(z - z_0)$$

together with the standard complex variable formulation (no continuation, that is to say) generated the fields associated with a point force acting at z_0 in an infinite solid. On the real line, these forces induce stresses [5]

$$\sigma_{22} - i\sigma_{12} = \lim_{x_2 \rightarrow 0} \left\{ \Omega'_0(z) + \overline{\Omega'_0(z)} + z \overline{\Omega''_0(z)} + \overline{\omega'_0(z)} \right\}$$

We must therefore superpose an additional solution $\Omega_1(z)$ which generates equal and opposite tractions on the surface of the half-plane. We could apply the procedure outlined in the preceding subsection to do this, but it is quicker to get the solution directly. Suppose that the corrective solution is to be generated by a potential $\omega_1(z)$, using the stress continuation discussed earlier. Then

$$\Omega_{1+}(z) - \Omega_{1-}(z) = -\Omega'_0(x_1) - \overline{\Omega'_0(x_1)} - x_1 \overline{\Omega''_0(x_1)} - \overline{\omega'_0(x_1)}$$

Now, observe that $\Omega_0(z)$ is analytic in R_- , while $z \overline{\Omega'_0(\bar{z})} + \overline{\omega_0(\bar{z})}$ is analytic in R_+ .

We may therefore satisfy the boundary conditions by setting

$$\Omega_1(z) = \begin{cases} -z \overline{\Omega'_0(\bar{z})} - \overline{\omega_0(\bar{z})} & z \in R_+ \\ \Omega_0(z) & z \in R_- \end{cases}$$

This solves our problem, but it is inconvenient to have part of the solution expressed using the standard complex variable representation, while the corrective term is expressed using the formulation based on stress continuation across the real axis. We can write the correction in the standard form by computing $\omega_1(z)$. Recall that

$$\omega(z) = -\overline{\Omega(\bar{z})} - z \overline{\Omega'(z)} \quad z \in R_+$$

so that

$$\omega_1(z) = -\overline{\Omega_0(z)} - z (-z \overline{\Omega''_0(\bar{z})} - \overline{\Omega'_0(\bar{z})} - \overline{\omega'_0(\bar{z})}) \quad z \in R_+$$

(Note that whenever we evaluate $\Omega_1(\zeta)$ we need to decide whether its argument lies in R^+ or R^- . Since z lies in R^+ , \bar{z} is in R^- .)

Then, finally, we may write

$$\omega(z) = \omega_0(z) + \omega_1(z) \quad \Omega(z) = \Omega_0(z) + \Omega_1(z)$$

giving

$$\begin{aligned} \Omega(z) &= \Omega_0(z) - z \overline{\Omega'_0(\bar{z})} - \overline{\omega_0(\bar{z})} \\ \omega(z) &= \omega_0(z) + z \overline{\omega'_0(\bar{z})} - \overline{\Omega_0(\bar{z})} + z \overline{\Omega'_0(\bar{z})} + z^2 \overline{\Omega''_0(\bar{z})} \end{aligned}$$

Stresses and displacements should be evaluated using the standard representation

$$\begin{aligned} 2\mu D &= (3 - 4\nu)\Omega(z) - z \overline{\Omega'(z)} - \overline{\omega(z)} \\ \sigma_{11} + \sigma_{22} &= 2(\Omega'(z) + \overline{\Omega'(z)}) \\ \sigma_{11} - \sigma_{22} + 2i\sigma_{12} &= -2(z \overline{\Omega''(z)} + \overline{\omega'(z)}) \end{aligned}$$

ensuring that both z and z_0 are in R^+ [5]

Exactly the same approach may be used to find the fields due to a dislocation near a free surface. Alternatively, using displacement continuation, we may compute the vibrational fields due to a dislocation near a rigid boundary. [5]

4.3 study of quantum mechanical relaxation processes

A major problem still confronting molecular simulations is how to determine time-correlation functions of many-body quantum systems. In this study the results of the maximum entropy (ME) and singular value decomposition (SVD) analytic continuation methods for calculating real time quantum dynamics from path integral Monte Carlo calculations of imaginary time

time-correlation functions are compared with analytical results for quantum mechanical vibrational relaxation processes. This system studied is an exactly solvable system: a harmonic oscillator bilinearly coupled to a harmonic bath. The ME and SVD methods are applied to exact imaginary-time correlation functions with various level of added random noise, and also to imaginary-time data from path integral Monte Carlo (PIMC) simulations. The information gathered in the present benchmark study is valuable for the application of the analytic continuation of PIMC data to complex systems [4]

4.3.1 MODEL SYSTEM

Let us consider an oscillator linearly coupled to a bath of harmonic oscillators. The Hamiltonian of the system is

$$H = H_{\text{osc.}} + H_{\text{bath}} + V_{\text{int.}} \quad (1)$$

where $H_{\text{osc.}}$ is the Hamiltonian of the free oscillator

$$H_{\text{osc.}} = \frac{p^2}{2m} + V(x), \quad (2)$$

where m is the reduced mass of the oscillator, and x and p are, respectively, the displacement of the oscillator from its equilibrium position and its conjugate momentum. The restoring force of the free oscillator is described by the potential $V(x)$. The Hamiltonian of the harmonic bath is the sum of the Hamiltonians of the component harmonic oscillators

$$H_{\text{bath}} = \sum_{\alpha} \left(\frac{p_{\alpha}^2}{2m_{\alpha}} + \frac{m_{\alpha}\omega_{\alpha}^2}{2} x_{\alpha}^2 \right), \quad (3)$$

where x_{α} is the coordinate of the α th oscillator, p_{α} its conjugate momentum, ω_{α} its equilibrium frequency, and m_{α} its reduced mass. The coupling between the oscillator and the bath is taken to be

$$V_{\text{int.}} = -x \sum_{\alpha} g_{\alpha} x_{\alpha}, \quad (4)$$

where the parameters g_{α} measure the degree of coupling of the oscillator with the α th normal mode of the bath. Let us assume also that the oscillator is coupled to an external radiation through its dipole $\mu(x) = q_0(X)$ that varies linearly with the displacement coordinate x . The bath is assumed not to be directly affected by the field. We are interested in the equilibrium dynamics of the oscillator, and in particular the quantum time autocorrelation function $\langle x(t)x(0) \rangle$ that ultimately determines the absorption of radiation by the system. The dipole absorption cross section $\sigma(\omega)$ is, in fact, given by

$$\sigma(\omega) = \frac{4\pi}{\hbar c} \omega (1 - e^{-\beta\hbar\omega}) I(\omega), \quad (5)$$

where the dipole spectral density $I(\omega)$ is defined as the Fourier transform of the dipole time autocorrelation function $q_0^2 \langle x(t)x(0) \rangle$

$$I(\omega) = q_0^2 \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle x(t)x(0) \rangle. \quad (6)$$

Thus the decay time of the envelope of the position correlation function, the vibrational dephasing, or energy relaxation time is related to the broadening of the absorption band of the oscillator. The parameter q_0 trivially scales by constant the dipole correlation function and the spectral function. In the following it will be omitted to simplify the notation.[4]

A. Classical treatment: the generalized Langevin equation

In a classical treatment of an oscillator embedded in a bath of harmonic oscillators, the dipole absorption cross section that describes the rate of energy absorption by the oscillator from an external oscillating radiation field is

$$\sigma(\omega)^{\text{cl.}} = \frac{4\pi\beta}{c} \omega^2 C_{\mu\mu}^{\text{cl.}}(\omega), \quad (7)$$

Where

$$\begin{aligned}
C_{\mu\mu}^{\text{cl.}}(\omega) &= \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle \mu(t) \mu(0) \rangle_{\text{cl.}} \\
&= \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle x(t) x(0) \rangle_{\text{cl.}}.
\end{aligned} \tag{8}$$

The autocorrelation function of the displacement of the oscillator can be obtained by solving the generalized Langevin equation

$$m\ddot{x}(t) = -\frac{\partial W[x(t)]}{\partial x} + \xi(t) - \int_0^t dt' \zeta(t-t') \dot{x}(t'), \tag{9}$$

where the time-dependent friction kernel $\zeta(t)$ is related to the spectral density of the bath modes

$$J_b(\omega) = \sum_{\alpha} \frac{g_{\alpha}^2}{2m_{\alpha}\omega_{\alpha}^2} [\delta(\omega + \omega_{\alpha}) + \delta(\omega - \omega_{\alpha})], \tag{10}$$

through a cosine transform

$$\zeta(t) = \int_{-\infty}^{+\infty} d\omega J_b(\omega) \cos(\omega t). \tag{11}$$

$W(x)$ is the potential of mean force. In the case of an harmonic bath it is given by

$$W(x) = V(x) - \frac{\zeta(0)}{2} x^2, \tag{12}$$

and $\zeta(t)$ is a Gaussian random force whose time autocorrelation function, by virtue of the fluctuation-dissipation theorem, is proportional to the friction kernel

$$\beta \langle \xi(t) \xi(0) \rangle = \zeta(t). \tag{13}$$

The generalized Langevin equation, Eq. (9) can be solved numerically by producing a set of realizations of the random force $\zeta(t)$ compatible with Eq. (13) and integrating Eq. 9 for each realization of the random force to obtain a set of trajectories $x(t)$. By averaging over the trajectories,[4]

the time autocorrelation $\langle x(t)x(0) \rangle_{cl}$ is finally recovered. For the particular case in which the potential $V(x)$ of the oscillator is also quadratic, $V(x) = m \omega^2 x^2 / 2$, a closed form for the absorption cross section can be derived

$$\sigma(\omega) = \frac{8\pi}{mc} \frac{\omega^2 \gamma'(\omega)}{[\tilde{\omega}^2 - \omega^2 + \omega \gamma''(\omega)]^2 + [\omega \gamma'(\omega)]^2}, \quad (14)$$

where $\tilde{\omega}^2 = \omega^2 - \zeta(0)/m$, and $[\gamma'(\omega)]$ and $[\gamma''(\omega)]$ are, respectively, the real and imaginary parts of the complex Laplace transform of the friction kernel, namely

$$\gamma(\omega) = \gamma'(\omega) + i\gamma''(\omega) = \frac{1}{m} \int_0^\infty dt e^{i\omega t} \zeta(t). \quad (15)$$

It can be shown that for a harmonic system the quantum mechanical and classical absorption cross sections coincide so that Eq. 14 is also valid when the oscillator and the bath modes are treated quantum mechanically. It follows, in particular, that the values vibrational dephasing and energy relaxation times are the same in either a classical or quantum mechanical treatment. [4]

4.3.2. QUANTUM TREATMENT: ANALYTIC CONTINUATION

It is extremely difficult to set up a direct numerical study of the real-time dynamics of an oscillator in a frictional bath in a quantum mechanical regime. In this paper we attempt to infer dynamical properties of the system through the analytic continuation of imaginary-time correlation functions. The legitimacy of such approach is ensured by the analyticity of quantum correlation functions. In particular, the real-time displacement correlation function $\langle x(t)x(0) \rangle, t \geq 0$, in Eq. (8) can be interpreted as the complex-time

displacement correlation function $\langle x(z)x(0) \rangle$, where z is a complex parameter, evaluated along the positive real time axis. On the same footing, the imaginary-time displacement correlation function $\langle x(-iT)x(0) \rangle$, $b \gg t > 0$, is interpreted as the complex-time displacement correlation function evaluated along the negative imaginary-time axis. The imaginary-time and real-time correlation functions are, thus, two equivalent representations of the same analytic function. One can be converted into the other by means of the analytic continuation operation. In performing the analytic continuation, it is useful to consider the spectral density $I(\omega)$ of $\langle x(t)x(0) \rangle$. By inverting Eq. (8) and by performing the replacement $t \rightarrow -iT$ where $t, T > 0$ we obtain

$$\langle x(-i\tau)x(0) \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{-\omega\tau} I(\omega), \quad (16)$$

The imaginary-time correlation function $\langle x(-iT)x(0) \rangle$ is, thus, the Fourier-Laplace transform of $I(\omega)$. Assuming $\langle x(-iT)x(0) \rangle$ is known for $0 < t < \beta\hbar$, the inversion of the integral equation 16 effectively completes the analytic continuation because $\langle x(t)x(0) \rangle$ is obtainable for real and positive t by a straightforward back Fourier transformation of $I(\omega)$

$$\langle x(t)x(0) \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} I(\omega). \quad (17)$$

It is convenient to perform the analytic continuation starting from the displacement imaginary-time correlation function of the position $R^2(-it) = \langle |x(-iT) - x(0)|^2 \rangle$. In terms of $R^2(-iT)$ and the dipole absorption cross section $\sigma(\omega)$ Eq. 16 Becomes

$$R^2(-i\tau) = \langle |x(-i\tau) - x(0)|^2 \rangle = \int_0^{+\infty} d\omega \sigma(\omega) K(\omega, \tau), \quad (18)$$

where the kernel function $K(\omega, t)$ is

$$K(\omega, \tau) = \frac{\hbar c}{4\pi^2} \frac{\cosh\left[\frac{\beta\hbar\omega}{2}\right] - \cosh\left[\beta\hbar\omega\left(\frac{1}{2} - \frac{\tau}{\beta\hbar}\right)\right]}{\omega \sinh\left(\frac{\beta\hbar\omega}{2}\right)}, \quad (19)$$

and the detailed balance relation $I(-\omega) = e^{-\beta\hbar\omega} I(\omega)$ has been used. The corresponding equation for $\langle |x(t) - x(0)|^2 \rangle$ can be easily derived by expressing Eqs. 18 and 19 in real time

$$\begin{aligned} R^2(t) &= \langle |x(t) - x(0)|^2 \rangle \\ &= \frac{\hbar c}{4\pi^2} \int_0^{+\infty} d\omega \sigma(\omega) \frac{1 - \cos(\omega t)}{\omega \tanh(\beta\hbar\omega/2)}. \end{aligned} \quad (20)$$

By differentiating twice Eq. 20 a relation between the real part of $\langle \dot{x}(t)\dot{x}(0) \rangle = \langle \mathbf{v}(t) \cdot \mathbf{v} \rangle$ and $\sigma(\omega)$ is obtained

$$\frac{8\pi^2 q^2}{\hbar c} \text{Re}[\langle \mathbf{v}(t) \cdot \mathbf{v} \rangle] = \int_0^{\hbar} d\omega \omega \sigma(\omega) \frac{\omega \cos \omega t}{\tanh \beta\hbar\omega/2}. \quad (21)$$

The imaginary-time correlation functions are readily available from path integral Monte Carlo (PIMC) simulations. The analytic continuation approach, therefore, has the clear advantage of avoiding the difficult task of following the dynamics of the system in real time. It suffers, however, from the fact that numerical analytic continuation is an illconditioned problem. In general, changes in the model system parameters produce small variations in the imaginary-time correlation functions but much larger variations in the real-time correlation functions. This means that by inverting Eq. (18) even extremely small statistical noise present in the imaginary-time correlation function can be amplified to such an extent that little can be said about the real-time dynamics of the system. By correctly handling of the statistical noise we can, at least, successfully identify those features of the absorption spectrum and of the real-time correlation function that are less affected by the statistical noise. [4]

4.4 Fractal theory or geometry

Fractal theory or geometry began in the seventeenth and more interest is given in the 18th, 19th and decades of the last century .

It give more interest to small matters and unlined Dynamics operations , it gives indication to scientific and technological problems which scientists ignore for a long time till before three years [2002] [7]

In the development countries more attention is given to insert the fractal theory or geometry in Arithmetic terminology to qualify the Arithmetic teachers .

Now and with the harmonise of Arithmetic and super science and art appeared theories and applies in the modern life and it is the fractal theory or geometry [7] .

4-4-1 : Fractal Theory or geometry conceptions and foundations

Mr . Beno Mandelbrot a polish country born and a French native and who employed recently to IBM company in America sat on the shore side in England enjoys the sea ... , it's waves , good weather sunny day . He deviated with hid sight to the other shore and he was wholly engaged with the protrusions and small gulfs and the rocky topographics inspire a poet or essay but twisted shore grow a problem in his mind many questions How long the England shore ? ... the shape of the twisted shore he motioned it by the self similarity , it is simply ((the shape consist of smaller shapes than it by various measures like a tree stem and its Branches , or vein with its parts or a river and it's tributaries ornamentation since the ancient times same as (Egyptian ang Islamic) and as he is An Aithmetic scientist trained through Borbaki Arithmetic school so he always mentioned kirtious Houssdorf , Jolia , kookhi and Beno interesting especially by the conceptions and the observed that included things with a self similarity for a few measures numbers [7]

He launched to define the self similar shapes (which is consist of smaller samples of it)

The fractal general definition ... (it is an orgeometry shape ((roughen or fracture) it can be divided in to parts every is at last reduction for the shape to various measures ... he went he wentback to the shore splits---it is fractals ! then he began an artmtric treatment to event new dimension for new fractals flew out.

Hausssdorf consider dimension to get new fractal through the treated function system ,these fractals in mathematics are similar to what in nature and he admire by(Jolya set) and that led him to invent strange and famous fractal and they named by his name [7]

In 1982 Mandbert wrote a book name Fractals . As a whole looking attentively to nature produce a new theory full of life and beauty and has a big effect on man life in dynamics , technical , biological and nature . This reflect [Horch ideas] about the human Arithmetic (made by some mathematicians) he says it is variable and effect the civilization development .

4.4.2 Self similarity and making some famous fractals

Self similarly considered a basic form . To orgeometry , some named it similar shape , we can divide the self similarity through the nature (statical in nature) and arts same as the mathematical tree .

4.4.3 Self similarity in nature

Do you want to now how to make some of the Fractal orgeometry ?

Biological , Scientists noticed the self similarity in Arithmetic , it is complicated and divided to smaller units for example as in plant [7]

Some examples of self similarity , the division of cell unit to two parts , then to eight units the complete of morula and stages of progress and gives smaller models from it in the fetus .

We can see four kinds of cells in man and animal they are :

1 . Epithelial

2 Tissue cells

3 .Muscle cells

4 . Nerve cells

The similar cells and the distinguished cells united to each other to consist the organs (kidneys – liver Heart) and these consist organ system then to organism

4.4.4 The repeated stage to produce Famous Fractals :

In fact you remember the repeated stages where you are making a successive approximation for doing roots equation according to (Newton stile).

$$X_{n+1} = \frac{Xn - f(Xn)}{f'(Xn)}$$

Applied by repeated stages

Or in complex form $Z_{n+1} = \frac{Zn - f(Zn)}{f'(Zn)}$

The result for every successive approximation start for the second till it reaches the best F value zero and exactly (Root equation) , so successive approximation is not just repeating but it is a repeat in (operation – applied rule) , to use out put all the repeat as in put in the next repeat .

This successive approximation we try to make it simple through Generator .

4.4.5 Koch snow flake curve :

You may enjoy yourself by arose opening and enjoy it's nice sell .. the more enjoyment will be if you see it from starting opening till it complies opening

The same when you following a jot from start to the end .

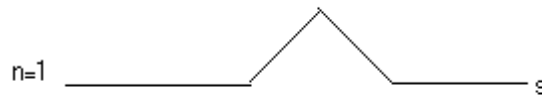
Koch the Sweden Erythematic scientist calls this (a long period fractal)

Now come let's enjoy snow flakes and how it consist from light to heavy snow .

The bud here is straight pieces s_0

length is =1

$n=0$ ————— S_0 **become**

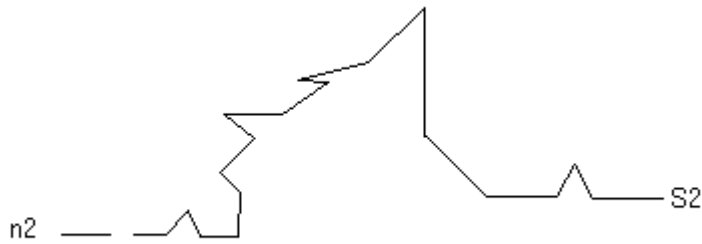


Generator in repeated n_2 exchange every strait piece for the first shape to the Generator shape then every of the four pieces and change the middle piece – that result (S_2) with 16 pieces every length .

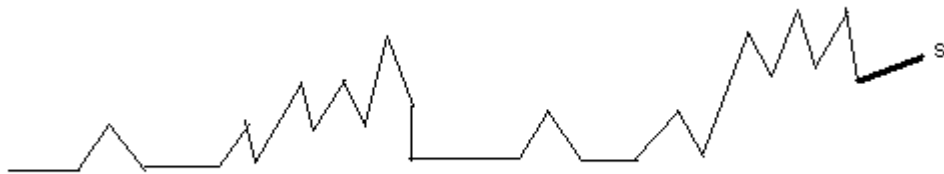
$$\delta = 3^{-2} = \frac{1}{9}$$

Start pieces $N = 4^2 = 16$

Length of the curve $L \left(\frac{1}{9} \right) = \left(\frac{4}{3} \right)^2 = \frac{16}{9}$



In the third stage n_2 change all straight pieces of the 16 in shape S_2 to Generator to result S_3



Observe the fractions are more specific when there is increase in the repeat and we come to snow flake [7]

So the Generator

$L(S) = (4/3)^n$ the length of the small straight piece $\delta = 3^{-n}$

Numerator of straight pieces [7]

$$N(\delta) = 4^n = 4^{\frac{\ln \delta}{\ln 3}}$$

It can be written $N(\delta) = \delta^{-D}$

Since

$$-D = \frac{\ln 4}{\ln 3}$$

$$D = \lim_{n \rightarrow \infty} \frac{\ln(4)^n}{\ln(3)^n}$$

$$D = \lim_{n \rightarrow \infty} \frac{\ln(4)^n}{\ln(3)^n} = \frac{n \ln 4}{n \ln 3}$$

Prophet Mohamed may God grant him peace and mercy says : ((think in the creatures of God and don't think in his being))

Also he said ((An hour of thinking is better than a year of worship))

In fact thinking and praise are interchanged .

A mathematical conception is a nearest approach and most honest science which we can reach with it to what ALLA Almighty said :

((سنريهم آياتنا في الآفاق وفي انفسهم حتى يتبين لهم انه الحق))

(صدق الله العظيم)

The creation of man in the Holly Koran and in modern Arithmetic is of important questions which must make scientists to answer through

((Amathematical science and Holly Koran))

We pray to Alla to rum wisdom and knowledge in our hearts and minds (Ammin)[7]

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