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Boundary Element Solutions For Some Different Temperature Distributions

A Thesis Submitted in Fulfillment

For the Degree of M.Sc. in Mathematics

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DEDICATION

To my family

My wife and my children

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Praise be to Allah, Lord of the world, Eternal Guardian of the heavens and earth. I praise him for his favours and ask him to increase his grace.

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ABSTRACT

The aims of this study which has been organized into five chapters is devoted to study different temperature distributions in mechanics of homogeneous and inhomogeneous media. A boundary element method (BEM) is derived for solving the two-dimensional steady-state and non steady-state temperature distributions. To check its validity, the proposed method is applied to solve some specific problems with known exact solutions. And it important for the design of steam and gas turbines, jet motors, rockets, high speed aircraft, nuclear reactors.

In chapter one, we discussed the numerical technique used in the thesis, where we have studied general BEM procedure for solving partial differential equation in one-dimensional. Then we generalized the BEM for solving two dimensional problems. Also, we discussed heat conduction in an undeformable body and principles of a boundary element technique, in the end of this chapter, we discussed the collocation method for solving a system of equations for the determination of the unknown boundary values as in our thesis's problems.

In chapter two, A boundary element method is presented for the numerical solution of a problem involving steady state two-dimensional heat conduction in homogeneous media. To reduce the differential equations to a system of linear algebraic equations, the temperature can be determined at any desired point in the interior of the solution domain. Numerical results obtained by using the boundary element method agree quite well with the exact solutions.

In chapter three, A two-dimensional problem which requires determining the non-steady temperature distribution in a homogeneous media. It is solved numerically using a dual-reciprocity boundary element method. Numerical results are obtained for specific test problem agree well with the exact solution.

In chapter four, a dual-reciprocity boundary element method is proposed for solving the two-dimensional steady-state temperature distribution in non-homogeneous media. Numerical results are obtained for specific test problem agree well with the exact solution.

In chapter five, the boundary element method (BEM) is proposed for the numerical solution of the two-dimensional non steady-state temperature distribution in non-homogeneous media. The physical solution is recovered through the use of a numerical technique of dual-reciprocity BEM. Such a method of solution is used to solve a specific problem which has a known exact solution. The numerical results obtained agree well with the exact solution.

Then the present study concludes generally that the boundary element method is more suitable for study of different temperature distributions.

مستخلص البحث

تهدف هذه الرسالة الى ايجاد الحلول العددية لبعض التوزيعات الحرارية المختلفة في الأوساط متباينة الخواص في جميع الأتجاهات (Anisotropic) وذلك باستخدام طريقة عنصر الحد (Boundary Element Method) مقارنة مع الحل المضبوط وهذه التوزيعات ذات أهمية كبيرة في صناعة الطائرات والصواريخ والتوربينات (Turbines) والمحركات الحرارية (Heat Engines) والمفاعلات النووية (Nuclear Reactors). تضمن البحث خمسة فصول

كان الغرض من الفصل الأول هو دراسة أساسيات التكنيك العددي المستخدم في الرسالة وتم توضيح مخطط عام للتكنيك في بعد واحد (One-dimensional) بحيث يمكن تعميمه في بعدين (Two-dimensional) وكذلك تم اشتقاق معادلة التوصيل الحراري من خلال قانوني الديناميكا الحرارية لنتوصل للتوزيعات الحرارية التي تم تناولها في فصول الرسالة كما تم توضيح طريقة التجميع (Collocation Method) التي تستخدم من خلال تطبيق طريقة عنصر الحد (Boundary Element method) المستخدمة في حل التوزيعات الحرارية المختلفة والتي تم تناولها بفصول الرسالة.

أما الفصل الثاني تضمن دراسة توزيع حراري مستقر في الأجسام المتجانسة (Homogeneous) في بعدين (Two-dimensional) وتم ايجاد حل لهذا التوزيع باستخدام طريقة عنصر الحد والتي تتطلب فقط حد نطاق الحل لتجزئته واختزلت التوزيع الذي ندرسه الى مجموعة من المعادلات الجبرية الخطية معاملاتها بسيطة في الشكل وسهلة الحساب لذا أمكننا تطبيقها بكفاءة على جهاز الحاسوب وكحالة خاصة امكن ايجاد قيم عددية لمثال محدد وتمت مقارنتها بالقيم المضبوطة ووجدنا انها تتفق جيدا معها.

تضمن الفصل الثالث دراسة توزيع حراري غير مستقر في الأجسام المتجانسة (Homogeneous) في بعدين (Two-dimensional) وتم ايجاد حل لهذا التوزيع باستخدام طريقة عنصر الحد والتي تتطلب فقط حد نطاق الحل لتجزئته واختزلت التوزيع الذي ندرسه الى مجموعة من المعادلات الجبرية الخطية لا تتضمن مجاهيل عند النقاط الداخلية لنطاق الحل ومن ثم تم الحصول على قيم عددية للحرارة عند نقط مختلفة داخل نطاق الحل ووجدناها تتفق جيداً مع قيم الحل المضبوط .

وكان الغرض من الفصل الرابع هو دراسة توزيع حراري مستقر داخل وسط حراري غير متجانس (Non-homogeneous) في بعدين (Two-dimensional) وامكن الحصول على التدفق الحراري داخل الوسط باستخدام طريقة عنصر الحد تحت شروط حدية مناسبة ووجدنا ان الحل العددي يتفق مع الحل المضبوط.

وتم في الفصل الخامس دراسة توزيع حراري غير مستقر في بعدين في وسط حراري غير متجانس (Non-homogeneous) في بعدين (Two-dimensional) وطبقنا طريقة عنصر الحد تحت شروط حدية وابتدائية مناسبة وامكن الحصول على التوزيع الحراري داخل الوسط ووجدنا ان الحل العددي يتفق مع الحل المضبوط.

و نظرا لدقة النتائج التي حصلنا عليها بواسطة طريقة عنصر الحد (Boundary Element) والتي تتفق مع الحل المضبوط فان طريقة عنصر الحد هي أفضل الطرق العددية لحل مسائل التوزيعات الحرارية المختلفة (Temperature Distributions) حيث تتطلب فقط تجزئة الحد وذلك مقارنة بطريقة الفروق المحدودة (Finite Difference) وطريقة العنصر المحدود (Finite Element) واللتان يتطلبن تجزئة السطح ككل.

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NOMENCLATURE

k	thermal conductivity.
k_{ij}	constant thermal conductivity.
q	heat flux.
r,θ	polar coordinate system.
τ	time
T	temperature.
x, y	Coordinates of Cartesian system.
ξ_i	ξ - location of the source point (load point).
η_i	η - location of the source point.
δ	Dirac delta function.
ε	sifting deviation.
Γ	domain boundary.
$d\Gamma$	surface element.
Ω	domain of the problem.
$d\Omega$	volume element.
S	the entropy.
$S^{(i)}$	entropy production rate.
$S^{(r)}$	external entropy input rate.
ρ	density.
ρr	heat source density.
θ	polar angle (2-D).
t	time.
R	two-dimensional region.
D	simple closed curve.
C	closed curve.
α	positive constant.
f, g, h, u	functions.
n_x *and* n_y	components of a unit normal vector to the curve D.
ω	test function.
λ_{ij}	heat conductivity coefficients.
∇	gradient operator.
x_i	location of the source point.
y_i	location of the source point.

CONTENTS

INTRODUCTION

The aim and the problem of the thesis:

To measure the accurateness of boundary element method (BEM) as a mathematical technique (Approximation or Numerical) and its importance in solving the problems than the exact solution, and the comparison between boundary element method (BEM) and the exact solution.

The hypothesis:

For the problem of this thesis, we assumed that there a difference between the numerical results of our solution with boundary element and other results of the finite difference solution.

Recently, thermodynamics has undergone marked development in connection with important problems arising during the design of steam and gas turbines, jet motors, rockets, high speed aircraft, nuclear reactors, microelectronics etc.

Heat flow from the gas stream in heat engines, aerodynamic heating in high-speed aircraft, the heat given out by nuclear reactors, etc. all lead to the fact that the components in these machines operate under conditions of non-uniform, unsteady heating which change the physical and mechanical properties of the materials. There are then temperature gradients accompanying the non-uniform temperature distribution throughout the various components. Because of constraints, a non-uniform temperature distribution in a component having a complex shape usually gives rise to thermal stresses. It is essential to know the magnitude and effect of these thermal stresses when carrying out a rigorous design of such components. The thermal stresses alone and in combination with the mechanical stresses produced by the external forces can given rise to cracks and rupture in components containing brittle materials. In the general case, the change in temperature of a body is caused not only by heat transport from the surroundings but also by the process of deformation. When the rate of deformation is finite, thermo-mechanical effects of another nature are of importance, namely the generation and flow of heat within the body, the occurrence of associated elastic and thermal waves, thermoelastic dissipation of energy, etc (Fahmy[32]).

The deformation of a body is inseparably connected with a change of its heat content and therefore with a change of the temperature distribution in the body. A deformation of a body which varies in time leads to temperature changes, and conversely. The science which deals with the investigation of the above coupled processes is called thermoelasticity. Anisotropic material is a material having mechanical properties that are not the same in all directions at a point in a body of it. There are no planes of material symmetry. That is, the properties are a function of the orientation at a point.

Thermoelasticity describes the behavior of elastic bodies under the influence of nonuniform temperature fields. It represents, therefore, a generalization of the theory of elasticity. The constitutive equations, i.e., the equations characterizing the particular material, are temperature-dependent and include an additional relation connecting the heat flux in the body with the local temperature gradient. This relation, know in its simplest form as Fourier's law, determines the temperature distribution in the body. The temperature and stress fields in a solid body are (in general) coupled. However, for the usual heat transfer occurring in an unevently heated solid body as the result of external heat sources, the influence of the stresses and strains on the temperature distribution can be ignored. This enables us to calculate the temperature distribution in the body on the basis of a well-defined heat transfer without regard to the state of stress. In a solid the transfer of heat occurtos in virtue of heat conduction alone. This has molecular-atomic character and is not accompanied by any macroscopic movement. Heat transfer at the surface of a body can occur in three ways: heat conduction, convection or radiation. In the case of convection the heat exchange occurs by virtue of the motion of non-uniformly heated fluid (or gas) contiguous with the body. Moreover, convective heat exchange is understood to be the sum of the heat carried by the fluid particles and by heat conduction. Heat exchange by radiation (radiant heat exchange) takes place between bodies separated by a distance (or between different parts of a body) by means of electromagnetic waves. The equation of heat conduction necessary for the study of temperature fields in elastic bodies. In the theory of thermal stresses which goes back to the beginings of the theory of elasticity, the classical heat conduction equation was used, which does not contain any term represent the deformation of the body. By knowing the temperature distribution (from the solution of the heat conduction) the displacement equations of the theory of elasticity were solved. The latter known terms proportional to the temperature gradient. Thermoelasticity deals with a wide class of phenomena. It contains the generalized theory of heat conduction, the generalized theory of thermal stresses, describes the temperature distribution produced by deformation and finally it contains a description of the phenomenon of thermoelastic dissipation. The congitive merits of this theory are very large indeed. In spite of its mathematical complexity, thermoelasticity enables us to examine, deeper than before, the mechanism of the deformation and thermal processes occurring in elastic bodies (Brebbia[18]).

In the postwar years 2nd international war, there has been a rapid development of thermoelasticity, stimulated by various engineering sciences. A considerable, progress in the field of aircraft and machine structures, mainly with gas and steam turbines, and the emergence of new topic in chemical and nuclear engineering have given rise to numerous problems in which thermal stresses play an important and frequently roles. During the past two decades, widespread attention has been given to thermoelasticity theories which admit a finite speed for the propagation of thermal signals. In contrast to the conventional theories based on parabolic-type heat equation, these theories involve a hyperbolic-type heat equation and are referred to as generalized theories. Various researchers authors have formulated these generalized theories on different grounds. For example, Lord and Shulman [43] obtained a theory on the basis of a modified heat conduction law which involves heat-flux rate, and Green and Lindsay [36] developed a theory by including temperature-rate among the constitutive variables.

During the fifties and early sixties of 20^{th} the century many general algorithms were produced and analysed for the solution of standard partial differential equations. Since then the emphasis has shifted toward the construction of methods for particular problems having special features which defy solution by more general algorithms. This approach often necessitates a greater awareness of the different physical backgrounds of the problems such as free and moving boundary problems, shock waves, singular perturbations and many others particularly in the thermoelasticity.

Boundary element method (also known as boundary integral equation) has been successfully used in a variety of areas in engineering science, such as potential theory, elastostatics, elastodynamics, elastoplasticity, fracture, fluid mechanics, heat conduction, acoustics, electromagnetism and soil- or fluid-structure interaction (Divo,et al [28]).

Over recent decades, the boundary element method (BEM) has received much attention from researchers and has become an important technique in the computational solution of a number of physical problems. In common with the better-known finite element method (FEM) and finite difference method (FDM), the boundary element method is essentially a method for solving partial differential equations (PDEs) and can only be employed when the physical problem can be expressed as such. As with the other methods mentioned, the boundary element method is a numerical method and hence it is an important subject of research amongst the numerical analysis community. However, the potential advantages of the BEM have seemed so considerable that the strongest impetus behind its development has come from the engineering community, in its enthusiasm to obtain flexible and efficient computer-based solutions to a range of engineering problems.

The boundary element method (BEM) is an important technique in the computational solution of engineering and scientific problems. In applying the boundary element method, only a mesh of the surfaces is required, making it easier to use and often more efficient that the more common finite element method.

The boundary element method is derived through the discretisation of an integral equation that is mathematically equivalent to the original partial differential equation. The essential re-formulation of the PDE that underlies the BEM consists of an integral equation that is defined on the boundary of the domain and an integral that relates the boundary solution to the solution at points in the domain. The former is termed a boundary integral equation (BIE) and the BEM is often referred to as the boundary integral equation method or boundary integral method. Over the last twenty years the term boundary element method has become more popular. The other terms are still used in the literature however, particularly when researchers wish to refer to the overall derivation and analysis of the methods, rather than their implementation or applications. This study to check validity of a boundary element method (BEM) for solving the problems; numerical results are given and compared with the exact solutions, and also with finite difference method, (Brebbia et al[21]).

An integral equation re-formulation can only be derived for certain classes of PDE. Hence the BEM is not widely applicable when compared to the near-universal adaptability of the finite element and finite difference method. However, in the cases in which the boundary element method is applicable, it often results in a numerical method that is easier to use and more computationally efficient than the competing methods.

The advantages in the boundary element method arises from the fact that only the boundary (or boundaries) of the domain of the PDE requires sub-division. (In the finite element method or finite difference method the whole domain of the PDE requires discretisation.) Thus the dimension of the problem is effectively reduced by one, for example an equation governing a three-dimensional region is transformed into one over its surface. In cases where the domain is exterior to the boundary, as it is in potential flow past an obstacle, the extent of the domain is infinite and hence the advantages of the BEM are even more striking; the equation governing the infinite domain is reduced to an equation over the (finite) boundary.

The importance of BEM is unique amongst numerical methods and is a direct consequence of three factors:

i) The friendliness and openness of the BEM Community and its ability to continue to grow by attracting researchers all the time.

ii) The Boundary Element Method was originally developed as a technique for engineers rather than a pure mathematical technique. This meant that the major motivation behind the method was to reduce the dependency of the analysis on the definition of meshes. This motivation allowed the method to expand naturally, into new areas such as Dual Reciprocity, Complex Variable that will be used in this thesis and other Mesh Reduction Techniques.

iii) The importance that BEM attached, right from the beginning, to produce industrial application tools. Complex mathematics was seen as subordinate to the needs of the practising engineer. The aim was to produce user-friendly codes which were seemingly effortless to use, while hiding inside very complex calculations, (Divo and Kassab [27]).

Abd-Alla [1,2,3,4] studied thermal stresse problems. Also, Abd-Alla and Ahmed [7] studied effect of initial stress overlying semi infinite medium. Clements [26] studied thermal stress in an anisotropic elastic half-space. Chang, et al. [25] used fundamental Green's functions for solving heat conduction equation in anisotropic media. Yaghoubi, et al. [50] studied a boundary element modeling for two-dimensional transient heat conduction, Wang, et al. [49] applied a dual reciprocity boundary element approach for the problems of large deflection of thin elastic plates. Karami and Hematiyan [38] studied a boundary element method of inverse non-linear heat conduction analysis with point and line heat sources. El-Naggar, et al. [30] used explicit difference scheme to obtain thermal stresses in a non-homogeneous media. Kögl and Gaul [41] used a boundary element method for anisotropic coupled thermoelasticity. Abd-Alla, et al. [9] studied thermoelastic stresses in non-homogeneous anisotropic media. Matsumoto, et al. [45] studied a simple technique for efficient evaluations of boundary integrals of time-harmonic elastodynamic BEM analyses for anisotropic solids. Fahmy [31, 32] studied an inhomogeneous anisotropic elastic material by using BEM.

The Complex Variable Boundary Element Method or CVBEM is a generalization of the Cauchy integral formula into a boundary integral equation method or BIEM. This generalization allows an immediate and extremely valuable transfer of the modeling techniques used in real variable boundary integral equation methods (or boundary element methods) to the CVBEM. Consequently, modeling techniques for dissimilar materials, anisotropic materials, and time advancement, can be directly applied without modification to the CVBEM, (Ang. et al [15]).

An extremely useful feature offered by the CVBEM is that the produced approximation functions are analytic within the domain enclosed by the boundary problem and, therefore, exactly satisfy the two-dimensional Laplace equation throughout the problem domain. Another feature of the CVBEM is the integrations of the boundary integrals along each boundary element are solved exactly without the need for numerical integration. Additionally, the error analysis of the CVBEM approximation functions is workable as easy-to-understand the concept of relative error. A sophistication of the relative error analysis is the generation of an approximative boundary upon which the CVBEM approximation function exactly solves the boundary conditions of the boundary value problem (of the Laplace equation), and the goodness of the approximation is easily seen as a closeness-of-fit between the approximative and true problem boundaries. This numerical approach can then be used to develop solutions for potential problems which occur in engineering applications, or to aid in numerically calibrating and verifying domain method numerical models (e.g. finite element or finite difference methods) of steady state diffusion type problems.

The dual-reciprocity boundary element method (DRBEM) was originally introduced by Brebbia and Nardini [19] and Patridge and Brebbia [47] for the numerical solution of dynamic problems in solid mechanics. The method has now been successfully applied to solve a wide range of problems in engineering.

Recently, Divo and Kassab [27, 28] have introduced a new technique for the development of a boundary integral equation for heat conduction in heterogeneous isotropic media.

This thesis devoted to study different temperature distributions for non-homogeneous anisotropic and isotropic medium. Boundary element method is considered to solve these problems under suitable initial and boundary conditions. This thesis consists of five chapters as formerly summarized.

CHAPTER (I)

(The basic concepts of boundary element method)

1.1 Boundary element Method for one dimensional problem

In this section we will consider a simple one dimensional (1-D) example to show how a differential equation can be transformed to the boundary by means of the method of weighted residuals. In this example, we will explain the basic steps that we will use in the derivation of the 2-D boundary element formulations, in order to understand the principles of the boundary element method.

The physical problem is usually described by the partial differential equation. To obtain an integral equation, which in many respects is easier to handle, we employ the technique of weighted residuals. In the resulting integral equation, the differential operator acts on the unknown field variable u. Now, by employing integration by parts, we can reduce the order of the differential operator acting on u step-by-step, until no more partial derivatives of u appear under the integral. In the process, we obtain a series of boundary integrals, and the domain integral now contains partial derivatives of the weighting function ω, i.e , we have shifted the differential operator from the field function to the weighting function. Now, by choosing the fundamental solution as the weighting on, we can eliminate the domain integral containing the field function. The resulting representation formula no longer contains any unknown variables in the domain integral. In 2-D, we have to discretise this equation, and finally obtain a system of equations. In order to obtain the unknown boundary values, we solve the system of equations by inserting the prescribed boundary conditions.

Now let us have a look at the inhomogeneous differential equation

$$\underbrace{\frac{d^2u}{dx^2}+u}_{\tau u}=-x \tag{1.1.1}$$

which is defined in the domain $0 \le x \le 1$. We prescribe the (homogeneous) boundary conditions $u=\bar{u}$ on the boundary Γ, which in the case of the 1-D example degenerates to two points:

$$\bar{u}_1 = u(x=0)=0 \quad \text{and} \quad \bar{u}_2 = u(x=1)=0. \tag{1.1.2}$$

According to (1.1.2), the differential operator is given by

$$\tau = \frac{d^2}{dx^2} + 1. \tag{1.1.3}$$

To transform (1.1.1) to the boundary, we proceed with the following steps.

Step 1: Weighting the differential equation with a test function ω.
This leads to the integral equation

$$\int_0^1 \left(\frac{d^2 u}{dx^2} + u + x \right) \omega dx = 0 \tag{1.1.4}$$

Step 2: Carrying out integration by Parts ($\int u'\omega dx = [u\omega] - \int u\omega' dx$) according to the order of the differential equation operator τ. In our example, the first integration by parts yields

$$\int_0^1 \frac{d^2 u}{dx^2} \omega dx = \left[\frac{du}{dx} \omega \right]_0^1 - \int_0^1 \frac{du}{dx} \frac{d\omega}{dx} dx \ , \tag{1.1.5}$$

and with the second integration by parts, we obtain from (1.1.4)

$$\underbrace{\int_0^1 \left(\frac{d^2 \omega}{dx^2} + \omega \right) u dx}_{\tau^* \omega} + \int_0^1 x \omega dx + \underbrace{\left[\frac{du}{dx} \omega \right]_0^1}_{[u'\omega]_0^1} - \underbrace{\left[u \frac{d\omega}{dx} \right]_0^1}_{[u\omega']_0^1} = 0. \tag{1.1.6}$$

The boundary terms $[*]_0^1$ now contain both known boundary values $(\bar{u}_1 = \bar{u}_2 = 0)$ and the unknown boundary values $u'(0)$ and $u'(1)$. Since the boundary values for u are given those for u' are unknown, we have a simple boundary-value problem.

If we had values of both u and u' as unknowns, we would speak of a mixed boundary-value problem. In addition, we see that the adjoint differential operator $\tau^* \equiv \tau$.in this case, we call the differential operator self-adjoint.

Step 3: Next we choose the fundamental solution u^* of τ^* defined by

$$\tau^* u^* = \frac{d^2 u^*}{dx^2} + u^* = \delta(x - \xi) \tag{1.1.7}$$

as weighting function. In (1.1.7), ξ is the so-called load point, at which the point source is applied. By choosing $\omega = u^*$, we obtain, with the sifting property of the Dirac distribution,

$$\int_{-\infty}^{\infty} u(x)\delta(x,\xi)dx = u(\xi)$$

the first integral in (1.1.6) may be written as follows

$$\int_0^1 \left(\frac{d^2u^*}{dx^2} + u^* \right) u dx = \int_0^1 \delta(x,\xi)u(x)dx = u(\xi), \tag{1.1.8}$$

which leads us to the so-called representation formula

$$u(\xi) = -\int_0^1 xu^*(x,\xi)dx - \left[\frac{du}{dx} u^* \right]_0^1 + \left[u \frac{du^*}{dx} \right]_0^1 \tag{1.1.9}$$

that yields the values $u(\xi)$ inside the domain if the boundary solution is known.

Step 4: Now we have to determine the fundamental solution u^*. Our example is special case of the 1-D Helmholtz equation

$$\frac{d^2u^*}{dx^2} + \lambda^2 u^* = \delta(x-\xi), \tag{1.1.10}$$

With $\lambda = 1$. The fundamental solution of (1.1.10) is defined in the full space, and is given by

$$u^* = \frac{\sin(\lambda|x-\xi|)}{2\lambda}. \tag{1.1.11}$$

Step 5: Inserting the fundamental solution (1.1.11) and boundary conditions (1.1.2) into (1.1.9), we obtain the representation formula

$$\begin{aligned} u(\xi) &= -\int_0^1 \frac{x}{2} \sin|x-\xi| dx - \left[u'(x) \tfrac{1}{2} \sin|x-\xi| \right]_0^1 \\ &= -\int_0^1 \frac{x}{2} \sin|x-\xi| dx - \left(\frac{u'(1)}{2} \sin|1-\xi| - \frac{u'(0)}{2} \sin|-\xi| \right), \end{aligned} \tag{1.1.12}$$

whish gives the potential at a point ξ in the domain in terms of boundary variables only; the domain integral does not contain any unknowns and can therefore be regarded as a constant term.

Step 6: Now we place the load point ξ on the boundary – this dose not pose any problems in 1-D problems – and obtain two equations for the unknown boundary values $u'(0)$ and $u'(1)$. With $\xi = 0$ and $\xi = 1$ we obtain the equations

$$u(\xi = 0) = -\int_0^1 \frac{x}{2}\sin|x|dx - \frac{u'(1)}{2}\sin 1 + \frac{u'(0)}{2}\sin 0,$$
$$u(\xi = 1) = -\int_0^1 \frac{x}{2}\sin\underbrace{|x-1|}_{1-x}dx - \frac{u'(1)}{2}\sin 0 + \frac{u'(0)}{2}\sin 1.$$

(1.1.13)

Since the boundary values for u are known $(u(\xi = 0) = 0$ and $u(\xi = 1) = 0)$, we can solve the system (1.1.13) and obtain the solution

$$u'(0) = \frac{1}{\sin 1} - 1 \quad \text{and} \quad u'(1) = \frac{\cos 1}{\sin 1} - 1.$$

(1.1.14)

Step 7: The last step is the calculation of the values $u(\xi)$ inside the domain. By inserting (1.1.14) into (1.1.12), we obtain for the representation formula

$$u(\xi) = -\frac{1}{2}\left(\frac{\cos 1}{\sin 1} - 1\right)\sin|1-\xi| + \frac{1}{2}\left(\frac{1}{\sin 1} - 1\right)\sin\xi - \int_0^1 \frac{x}{2}\sin|x-\xi|dx.$$

(1.1.15)

From (1.1.15), we obtain for instance, the value $u(1/2)$ in the center of the interval

$$u(\tfrac{1}{2}) = -\frac{1}{2}\left(\frac{\cos 1 - \sin 1}{\sin 1}\right)\sin\frac{1}{2} + \frac{1}{2}\left(\frac{1}{\sin 1} - 1\right)\sin\frac{1}{2}$$
$$-\frac{1}{2}\int_0^{1/2} x\sin\left(\frac{1}{2} - x\right)dx - \frac{1}{2}\int_{1/2}^1 x\sin\left(x - \frac{1}{2}\right)dx$$
$$= \frac{\sin 0.5}{\sin 1} - \frac{1}{2} = 0.0697$$

(1.1.16)

We observe, by comparison with the analytical solution

$$u(x) = \frac{\sin x}{\sin 1} - x,$$

(1.1.17)

that the result in (1.1.16) is exact.

The expression boundary element formulation should not yet be used since we do not any discretisation in 1-D yields the exact solution might come as a surprise, since we have based the formulation upon a weighted residual statement which allows an error ε inside the domain and only demands that the integral average of the error be zero. Hence, despite the use of the weighted residual formulation, (1.1.12) yields the exact solution of the differential equation. This can be explained as follows: for every load point ξ_n inside the domain we obtain a different fundamental solution $u_n^* = u^*(x, \xi_n)$. Therefore, the weighted residual statement (1.1.4) with $\omega_n = u_n^*$ corresponds to a formulation with infinitely many linearly independent test functions, such that (1.1.4) can only vanish for all test functions ω_n if the expression in parentheses vanishes, i.e., if the differential equation is fulfilled exactly.

The procedure that we have described in the previous section can be formally generalized to two-dimensional problems as demonstrated by means of the generic differential equation

$$\tau u - b = 0, \tag{1.1.18}$$

which is defined in an arbitrary domain Ω. In (1.1.18), τ is an arbitrary differential operator with constant coefficients, u is the filed and b is an arbitrary source distribution in Ω. In our one example, we hade (cf.(1.1.1)) $\tau = \dfrac{d^2}{dx^2} + 1$ and $b(x) = -x$. The weighted form of (1.1.18) is now given by

$$\int_{x_1}^{x_2} (\tau u - b)\omega dx = 0. \tag{1.1.19}$$

In the multi-dimensional case, we obtain

$$\int_{\Omega} (\tau u - b)\omega d\Omega = 0. \tag{1.1.20}$$

Employing integration by parts according to the order of the differential operator, we obtain in 1-D

$$\int_{x_1}^{x_2} \tau^* \omega u dx + [Gu.S^*\omega]_{x_1}^{x_2} - [Su.G^*\omega]_{x_1}^{x_2} - \int_{x_1}^{x_2} \tau u \omega dx = 0, \tag{1.1.21}$$

Where G and S are boundary operators related to u, and G^*, S^* are the adjoint operators related to ω. Equation (1.1.6) is a special case of this general formulation with $G = \frac{d}{dx}$ and $S = 1$.

In two and three dimensions, we obtain a similar result by using integration by parts and Gauss theorem to reduce the domain integrals to boundary integrals:

$$\int_{\Omega} \tau^* \omega u d\Omega + \int_{\Gamma} (Gu.S^*\omega - Su.G^*\omega) d\Gamma - \int_{\Omega} \tau u \omega d\Omega = 0 . \tag{1.1.22}$$

Once again, by choosing the fundamental solution of the adjoint operator

$$\tau^* u^* = -\delta(x, \xi) \tag{1.1.23}$$

as the weighting function $(\omega = u^*)$, we can eliminate the first term in (1.1.22) by virtue of the sifting property of the Dirac distribution, and we obtain the representation formula

$$u(\xi) = \int_{\Gamma} (Gu..S^* u^* - Su.G^* u^*) d\Gamma - \int_{\Omega} b u^* d\Omega , \tag{1.1.24}$$

where the second domain integral has been replaced by (1.1.18).
The prescribed boundary conditions can also be generalized:

$$Gu = \overline{G} \qquad \text{on } \Gamma_G \ , \tag{1.1.25}$$

$$Su = \overline{S} \qquad \text{on } \Gamma_S \ , \tag{1.1.26}$$

where (1.1.25) describes the Dirichlet boundary conditions, and (1.1.26) the Neuman boundary conditions.

The representation formula (1.1.24) is only defined if ξ lies inside the domain. By moving the load point to the boundary in a special limiting process which will be described in detail in the following chapters – we obtain the so called boundary integral equation (BIE), in which all unknown field variables have been transformed to the boundary. The BIE is the starting point for the boundary element method: by discretising the boundary into finite elements on the boundary', we can approximate with the geometry and field variables, and by using this approximation with the BIE , we can set up a system of equations which contains only nodal values of u on the boundary. Solution of the system then yields the unknown boundary values, and, with the representation formula (1.1.24), we can obtain the solution inside the domain at any arbitrary point $\xi \in \Omega$.

1.1.1 Boundary Element formulation of Laplace's Equation

Now we will explain in detail the derivation of the BEM for Laplace's equation

$$u_{,ii} = 0 \quad \text{in} \quad \Omega \tag{1.1.27}$$

using the generic variable u for the potential and

$$q_i = u_{,i} \qquad \rightarrow \qquad q = u_{,i} n_i \tag{1.1.28}$$

for the flux. The results can then easily be applied to steady-state heat conduction, electrostatics , and other problems by identifying $T, \varphi,$ etc. with the generic potential u and the respective fluxes with the generic flux vector q_i. For heat conduction and electrostatics, for example we have

$$q_i = -\frac{1}{k} q_i^{th} \quad \text{and} \quad q_i = -\frac{1}{\varepsilon} q_i^{el}, \tag{1.1.29}$$

respectively.

The first step in the boundary element formulation consists of transforming the governing differential equation to an integral equation. As described above, this can be achieved by using the method of weighted residuals. Weighting Laplace's equation (1.1.27) with a test function ω, we obtain

$$\int_\Omega u_{,ii} \omega d\Omega = 0. \tag{1.1.30}$$

Next, we eliminate the partial derivatives of the potential u from the domain integral. This is achieved as follows: integration by parts of (1.1.30) leads to

$$\int_\Omega u_{,ii} \omega d\Omega = \int_\Omega (u_{,i} \omega)_{,i} d\Omega - \int_\Omega u_{,i} \omega_{,i} d\Omega, \tag{1.1.31}$$

and by applying the Gauss theorem (Ang [13]) to the first term on the right-hand side of (1.1.31) to replace the domain integral by a boundary integral, we obtain Green's first identity

$$\int_\Omega u_{,ii} \omega d\Omega = \int_\Gamma u_{,i} \omega n_i d\Gamma - \int_\Omega u_{,i} \omega_{,i} d\Omega. \tag{1.1.32}$$

to eliminate the remaining partial derivative $u_{,i}$ in the domain integral on the right-hand side, we have to again apply integration by parts and Gauss theorem. This yields

$$\int_{\Omega} u_{,ii}\omega d\Omega = \int_{\Gamma} (u_{,i}\omega - u\omega_{,i})n_i d\Gamma + \int_{\Omega} u\omega_{,ii} d\Omega \tag{1.1.33}$$

which is known as Green's second identity.

By substituting the differential equation (1.1.30) into (1.1.33), we eliminate the first domain integral and obtain

$$-\int_{\Omega} u\omega_{,ii} d\Omega = \int_{\Gamma} (u_{,i}\omega - u\omega_{,i})n_i d\Gamma \,. \tag{1.1.34}$$

1.1.2 Green's Representation Formula

The key point in the boundary element formulation is now the elimination of the remaining domain integral in (1.1.34), so that the subsequent discretisation needs to be applied only to the boundary Γ of the material body and not to its domain Ω. This is an advantage when compared to domain discretisation methods such as the finite element method (FEM) or the finite difference method (FDM) and can lead to important time-savings in the discretisation process and thus in the overall computational costs.

Using the Dirac distribution $\delta(x,\xi)$ by its sifting property $\int f(x)\delta(x,\xi)dx = f(\xi)$. This allows us to filter out a specific functional value $f(\xi)$ from an integral, thereby eliminating this integral. We will now employ this property to eliminate the domain integral in (1.1.34) by choosing

$$\omega_{,ii} := -\delta(x,\xi) \,, \tag{1.1.35}$$

which

$$-\int_{\Omega} u\omega_{,ii} d\Omega = u(\xi) \,. \tag{1.1.36}$$

For convenience, whenever the field point vector x_i and load vector ξ_i appear as arguments of functions, they will be written in the following as x and ξ, i.e., $f(x,\xi) \equiv f(x_i,\xi_i)$.

The function ω as defined in (1.1.35) is a so-called fundamental solution. In general, a fundamental solution u^* of the differential operator τ^* is defined as a solution of the equation

$$\tau^* u^* = -\delta(x,\xi) \tag{1.1.37}$$

in the full space Ω^∞, where the minus sign in front of the Dirac distribution is used for convenience.

Employing now as test function ω the fundamental solution u^*, we obtain from (1.1.34)

$$u(\xi) = \int_\Gamma (q(x)u^*(x,\xi) - u(x)q^*(x,\xi))d\Gamma ,$$

(1.1.38)

where $q^* := u^*_{,i} n_i$ is the fundamental solution for the flux. Equation (1.1.38) is a so-called representation formula, which in this particular case is also known as Green's representation formula and is valid for 2-D potential problems. The representation formula allows us to calculate unknown values of the potential u inside the domain $(\xi \in \Omega)$ when the boundary solution of the problem (potential u and flux q) is known.

Now in 2-D, the fundamental solution of the Laplace's operator as defined in (1.1.35) is given by (Gual, et al. [34])

$$u^*(x,\xi) = -\frac{1}{2\pi}\ln|x_i - \xi_i| = -\frac{1}{2\pi}\ln r ,$$

(1.1.39)

$$q^*(x,\xi) = u^*_{,i} n_i = -\frac{1}{2\pi r} r_{,i} n_i = -\frac{1}{2\pi|x_i - \xi_i|^2}(x_i - \xi_i)n_i ,$$

(1.1.40)

1.2 Heat Conduction

In this section, we will derive the equations of heat conduction in an undeformable body.

1.2.1 First Law of Thermodcynamics

In the absence of mechanical forces, the first law of thermodynamics states that the time rate of change $\dot{U}$ of the interval energy is equal to the rate of external heat supply:

$$\dot{U} = \frac{dQ}{dt} ,$$

(1.2.41)

where the symbol d is again employed to make clear that the supplied heat Q is a process variable and not a state variable, and therefore dose not possess a total differential. The heat rate is given by

$$\frac{dQ}{dt} = \int_{\Omega} \rho r d\Omega - \int_{\Gamma} q_i n_i d\Gamma$$

(1.2.42)

and consists of two parts: the heat generated inside the volume element $d\Omega$ is described by the heat source density ρr , and the heat supplied over the surface element $d\Gamma$ is described by the heat flux vector q_i. Possible mechanisms that generate heat in the volume element are electric Joule heating, absorption of thermal radiation, or chemical and reactions (Gaul, et al. [34]). The boundary integral over the heat flux vector has a negative sign since an outward heat flux means that the system loses energy. Therefore we obtain for the first law

$$\frac{D}{Dt}\int_{\Omega} \rho u d\Omega = \int_{\Omega} \rho r d\Omega - \int_{\Gamma} q_i n_i d\Gamma .$$

(1.2.43)

By employing the generalised Gauss theorem, the surface integral can be converted to a volume integral and we obtain the differential form of the first law

$$\rho \dot{u} = \rho r - q_{i,i} .$$

(1.2.44)

1.2.2 Second Law of Thermodynamics

We know from experience that heat cannot flow from lower temperatures 'by itself'. This phenomenon cannot be described by the first law of thermodynamics, which is an energy balance and as such poses no restrictions on the direction of heat exchange processes. To describe the direction and irreversibility of thermodynamic processes, we introduce a new extensive state variable, the entropy S .

The second law of thermodynamics now states that the time rate of change $\dot{S}$ of the entropy is given by the sum of an external entropy input rate $\dot{S}^{(r)}$ and an entropy production rate $\dot{S}^{(i)}$:

$$\dot{S} = \dot{S}^{(r)} + \dot{S}^{(i)},$$

(1.2.45)

where the entropy production rate cannot be negative:

$$\dot{S}^{(i)} \geq 0 .$$

(1.2.46)

For reversible processes, $\dot{S}^{(i)} = 0$, while irreversible processes are characterized by a positive entropy production rate.

For a continuous system, the entropy input rate $\dot{S}^{(r)}$ is given by (Carslaw and Jaeger [23])

$$\dot{S}^{(r)} = \int_{\Omega} \frac{\rho r}{T} d\Omega - \int_{\Gamma} \frac{q_i n_i}{T} d\Gamma ,$$

(1.2.47)

where T is the absolute temperature. With this, we obtain, as an alternative formulation of the second law of thermodynamics, the so-called Clausius-Duhem inequality (Gaul. et al [34]).

$$\frac{D}{Dt} \int_{\Omega} \rho s d\Omega \geq \int_{\Omega} \frac{\rho r}{T} d\Omega - \int_{\Gamma} \frac{q_i n_i}{T} d\Gamma .$$

(1.2.48)

By using Gauss integral theorem, this yields the local form

$$T\rho\dot{s} - \rho r + q_{i,i} - \frac{q_i T_{,i}}{T} \geq 0 .$$

(1.2.49)

With the first law (1.2.4), we can eliminate the heat source density ρr and obtain

$$-\rho(\dot{u} - T\dot{s}) - \frac{q_i T_{,i}}{T} \geq 0 .$$

(1.2.50)

When the specific entropy s is chosen as the independent state variable, it follows that $u = u(s)$ and thus

$$\dot{u} = \frac{\partial u}{\partial s} \dot{s} ,$$

(1.2.51)

which yields

$$-\left(\frac{\partial u}{\partial s} - T \right) \rho\dot{s} - \frac{q_i T_{,i}}{T} \geq 0 .$$

(1.2.52)

Since s - and $\dot{s}$ - is arbitrary, we have

$$T = \frac{\partial u}{\partial s} ,$$

(1.2.53)

i.e., the absolute temperature T and the specific entropy s are energetically conjugate state variables. Since the absolute temperature is always positive $(T > 0)$, it follows from (1.2.52) that

$$q_i T_{,i} \leq 0 ,$$

(1.2.54)

which means that heat can only flow from higher to lower temperatures. Thus the second law of thermodynamics poses restrictions on the direction of heat transfer processes as previously demanded. The simplest relation that fulfils (1.2.54) is Fourier's law of heat conduction

$$q_i = -k_{ij} T_{,j} \ , \tag{1.2.55}$$

where k_{ij} is the tensor of thermal conductivity, which is usually taken to be symmetric. This does not follow from thermodynamics but from the Onsager relations, obtained from considerations at the microscopic level, and is well confirmed experimentally (Lesnic, et al. [42]). Substitution of (1.2.55) into the second law (1.2.54) yields

$$k_{ij} T_{,i} T_{,j} \geq 0 , \tag{1.2.56}$$

thus the thermal conductivity tensor k_{ij} has to be positive semi-definite to comply with the second law.

With $\dot{u} = T\dot{s}$, we finally obtain, from the first law (1.2.44), the local entropy balance

$$\rho T\dot{s} = \rho r - q_{i,i} \tag{1.2.57}$$

1.2.3 Field Equations of Heat Conduction

We can now derive the field equations of heat conduction from the differential from (1.2.44) of the first law. To this end, we choose the temperature T as the independent state variable, so that $u = u(T)$ and thus

$$\dot{u} = c(T)\dot{T} , \tag{1.2.58}$$

where

$$c(T) := \frac{\partial u(T)}{\partial T} \tag{1.2.59}$$

is the heat capacity of the material. With (1.2.59) and Fourier's law (1.2.55), we obtain from the first law (1.2.44) the heat conduction equation

$$c(T)\rho\dot{T} = (k_{ij} T_{,j})_{,i} + \rho r , \tag{1.2.60}$$

which is a parabolic diffusion-type equation. For a homogeneous body with constant thermal conductivity k_{ij} and temperature-independent heat capacity, this simplifies to

$$c\rho\dot{T} = k_{ij} T_{,ij} + \rho r . \tag{1.2.61}$$

For an isotropic medium, $k_{ij} = \delta_{ij} k$, which yields for the heat conduction equation

$$\dot{T} = aT_{,ii} + \frac{r}{c}$$

(1.2.62)

with the thermal diffusivity $a := k/(c\rho)$. A further simplification can be achieved by assuming steady-state heat conduction, which is described by Poisson's equation

$$T_{,ii} = -\frac{\rho r}{k}.$$

(1.2.63)

When heat sources are absent, this reduces further to Laplace,s equation

$$T_{,ii} = 0.$$

(1.2.64)

1.3.4 Boundary and Initial Conditions

For transient heat conduction described by (1.2.61), we have to know the initial temperature distribution

$$T(t=0) = T^0 \qquad \text{in } \Omega.$$

(1.2.65)

The boundary conditions in heat conduction can be classified as follows:
* Dirichlet boundary condition: the primary field variable (here the heat temperature T) is prescribed:

$$T = \overline{T} \qquad \text{on } \Gamma_T .$$

(1.2.66)

* Neumann boundary condition: the secondary field variable (here the heat flux q) is prescribed:

$$q = \overline{q} \qquad \text{on } \Gamma_q .$$

(1.2.67)

* Robin boundary condition: a function of the temperature and heat flux is prescribed:

$$f(T, q) = 0 \qquad \text{on } \Gamma_{Tq}.$$

(1.2.68)

For practical analyses, the following boundary conditions are of particular importance.

1.3 Boundary element method for 2-D Problem

While the formulation described in the previous sections for 2-D potential problems, the following steps – which include the derivation of the boundary integral equation and the discretisation process – depend upon the dimension of the problem. In the remainder of this thesis, we will deal exclusively with 2-D problems.

As noted before, the representation formula (1.1.38) returns the values of the potential u in the interior of the domain when the boundary solution is known. Hence, to obtain an equation that contains only boundary data, we have to move the load point ξ to the boundary. The resulting equation is called the boundary integral equation (BIE) and forms the basis of the subsequent discretisation process by the boundary element method.

The process of moving the load point to the boundary requires some care, since the sifting property of the Dirac distribution is not defined when the load point lies on the boundary:

$$\int f(x)\delta(x,\xi)d\Omega = \begin{cases} f(\xi) & \text{for} \quad \xi \in \Omega \\ 0 & \text{for} \quad \xi \notin \Omega, \xi \notin \Gamma \\ \text{undef.} & \text{for} \quad \xi \in \Gamma \end{cases} . \tag{1.3.69}$$

In the following, we will solve this problem by modifying the boundary in the vicinity of the lode point and then move the lode point to the boundary in a limiting process.

1.3.1 Classification of singularities in the BEM

In the direct BEM, we usually only have to deal with two types of singularity (see table 1.1)

$$\int_{-a}^{b} \frac{1}{x} dx = \ln|b| - \ln|-a|, \qquad a,b > 0 \tag{1.3.70}$$

as shown, the correct result is obtained. However, on second thoughts, we note that this must have been by chance, since the presence of the $1/x$-singularity in the integration interval has not been taken into account properly. The problem at $x = 0$ becomes apparent when trying to calculate the improper integral at $\int_{x=0}^{b} \frac{1}{x} dx$, which is undefined.

If we approach the singularity in (1.3.70) by a limiting process, we obtain

$$\int_{-a}^{b} \frac{1}{x} dx = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \left(\int_{-a}^{-\varepsilon_1} \frac{1}{x} dx + \int_{\varepsilon_2}^{b} \frac{1}{x} dx \right)$$
$$= \lim_{\varepsilon_1, \varepsilon_2 \to 0} (\ln \varepsilon_1 - \ln a + \ln b - \ln \varepsilon_2)$$
$$= \lim_{\varepsilon_1, \varepsilon_2 \to 0} \left(\ln \frac{\varepsilon_1}{\varepsilon_2} \right) + \ln b - \ln a .$$

(1.3.71)

We see that the result depends on ε_1 and ε_2 if they approach zero with different values. However, by choosing $\varepsilon_1 = \varepsilon_2 = \varepsilon$, we obtain

$$\ln \frac{\varepsilon}{\varepsilon} = \ln 1 = 0 ,$$

(1.3.72)

and thus (1.3.71) yields the correct result

$$\int_{-a}^{b} \frac{1}{x} dx = \ln \frac{b}{a} .$$

(1.4.73)

The integrand $f(x) = 1/x$ in (1.3.70) is strongly singular at $x = 0$, which means that its integral $F(x) = \int f(x) dx$ is singular at $x = 0$, too. The value of the integral as calculated with the limiting process in (1.3.71) is called a Cauchy principal value (CPV) of the strongly singular integral and is denoted by

$$\int_{-a}^{b} \frac{1}{x} dx .$$

(1.3.74)

In addition to the strongly singular integrands, we also encounter weakly singular integrands in the boundary element method. In constant to the strong singularity, the integral over a weakly singular integrand exists and is continuous at the singularity point. An example for this is the $\ln|x|$-function. At $x = 0$ the function is singular but the integral

$$\int \ln\left|\frac{1}{x}\right| dx = x \ln|x| - x + c$$

(1.3.75)

is continuous, which can be confirmed by applying the rule of l'Hosoital:[34]

$$\lim_{x \to 0} x \ln|x| = \lim_{x \to 0} \frac{\ln|x|}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0 .$$

(1.3.76)

Type	Property	2-D
weak singularity	integral is finite at singularity	$\ln r$
strong singularity	interpretation as Cauchy principal value	$\frac{1}{r}$

Table 1.1. Classification of singularities in the boundary element method

To move the load point to the boundary, we first modify the original boundary Γ, augmenting it by a small circular region with radius ε around the load point $\xi \in \Gamma$ as shown in figure 1.1. The modified boundary Γ' is then given by

$$\Gamma' = \Gamma - \Gamma_\varepsilon^* + \Gamma_\varepsilon, \tag{1.3.77}$$

so that

$$\Gamma = \lim_{\varepsilon \to 0} \Gamma'. \tag{1.3.78}$$

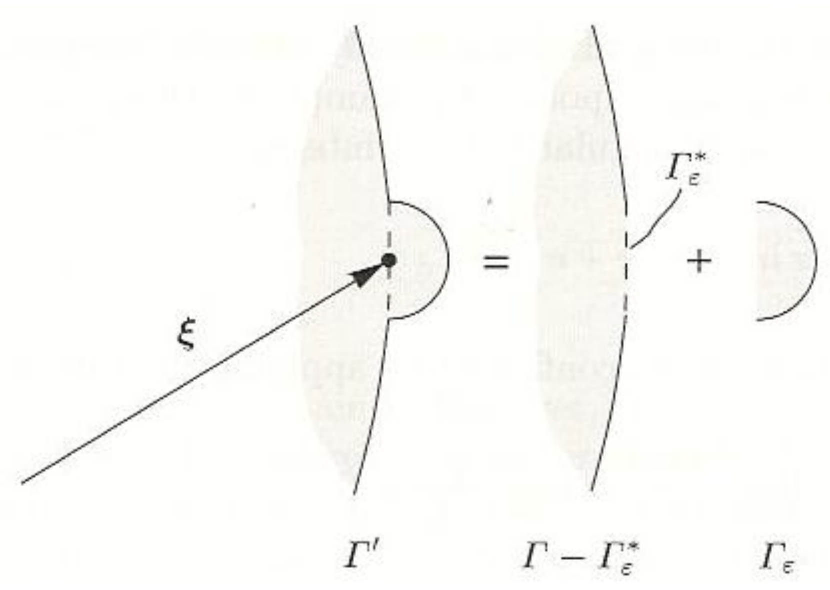


Figure 1.1. Boundary extension around lode point ξ

By this process, the load point ξ again comes lie inside the domain and the representation formula (1.1.38) remains valid.

As shown in figure 1.2, the line element $d\Gamma_\varepsilon$ along the boundary extension can be parametrised by

$$d\Gamma_\varepsilon = \varepsilon d\theta, \tag{1.3.79}$$

where

$$\varepsilon = |x_i - \xi_i| \tag{1.3.80}$$

is the Euclidean distance between the lode point ξ and the point x. With this, we can perform the limiting process when moving the load point to the boundary.

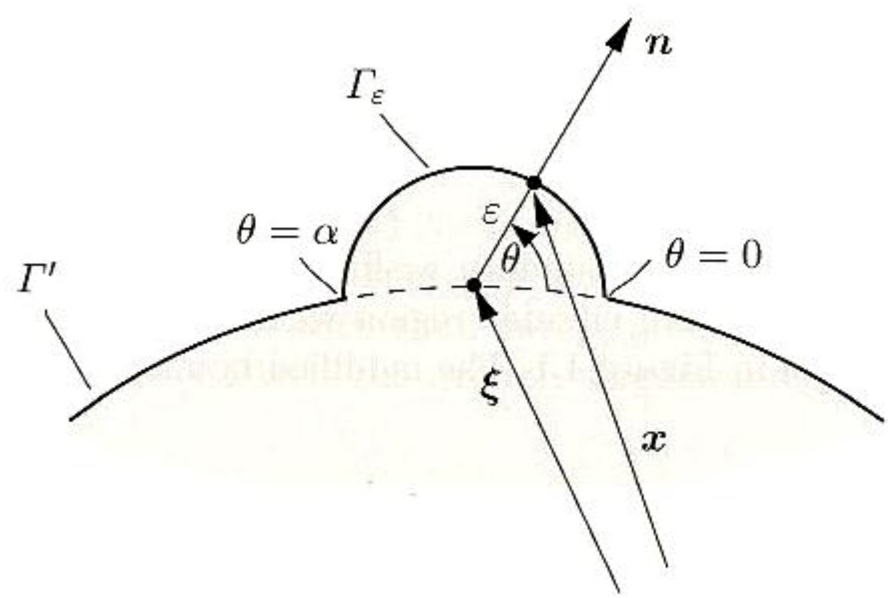


Figure 1.2 Geometry of the augmented boundary

(I) Weakly Singular Integral

Using the 2-D fundamental solution u^* given in (1.1.39), we obtain for the first term in (1.1.38)

$$\int_{\Gamma} q u^* d\Gamma = -\lim_{\varepsilon \to 0} \int_{\Gamma'} q \frac{\ln|x_i - \xi_i|}{2\pi} d\Gamma$$
$$= -\lim_{\varepsilon \to 0} \int_{\Gamma - \Gamma_\varepsilon^*} q \frac{\ln|x_i - \xi_i|}{2\pi} d\Gamma - \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} q \frac{\ln|x_i - \xi_i|}{2\pi} d\Gamma . \tag{1.3.81}$$

The first integral in (1.3.81) is weakly singular, so its calculation requires no special care. For the second integral, we obtain with (1.3.81), (1.3.82) and 1'Hospital rule

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} q\frac{\ln|x_i-\xi_i|}{2\pi}d\Gamma = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\theta=0}^{\alpha} q(\ln\varepsilon)\varepsilon d\theta$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\theta=0}^{\alpha} q\frac{(\ln\varepsilon)'}{(\frac{1}{\varepsilon})'}d\theta$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\theta=0}^{\alpha} -q\varepsilon d\theta = 0$$

(1.3.82)

Since the integral $\int_\Gamma qu^* d\Gamma$ is continuous at the (weak) singularity of u^*, the term in (1.3.82) over the boundary extension Γ_ε becomes zero for $\varepsilon \to 0$.

(II) Strongly Singular Integral

For the strongly singular integral in (1.1.38), we have

$$-\int_\Gamma uq^* d\Gamma = \lim_{\varepsilon\to 0}\int_{\Gamma'} u\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}d\Gamma$$

$$= \lim_{\varepsilon\to 0}\int_{\Gamma-\Gamma_\varepsilon^*} u\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}d\Gamma + \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} u\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}d\Gamma .$$

(1.3.83)

Since $\int_\Gamma uq^* d\Gamma$ is strongly singular, the integral over the modified boundary $\Gamma-\Gamma_\varepsilon^*$ represents its Cauchy principal value:

$$\lim_{\varepsilon\to 0}\int_{\Gamma-\Gamma_\varepsilon^*} uq^* d\Gamma = \int_\Gamma uq^* d\Gamma$$

(1.3.84)

For the second integral in (1.3.15), we obtain

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} u\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}d\Gamma = \lim_{\varepsilon\to 0}\int_{\theta=0}^{\alpha} u\frac{\varepsilon}{2\pi\varepsilon^2}\varepsilon d\theta = u(\xi)\int_{\theta=0}^{\alpha}\frac{1}{2\pi}d\theta$$

$$= \frac{\alpha}{2\pi}u(\xi) .$$

(1.3.85)

In contrast to (1.3.82), this term dose not vanish but remains finite, since the integrand is singular at $\varepsilon = 0$. The strongly singular integral $\int_\Gamma uq^* d\Gamma$ is therefore given by the sum of its Cauchy principal value and the contribution from (1.3.85).

By inserting the results for the weakly and strongly singular integrals into (1.1.38), we obtain the boundary integral equation

$$u(\xi) = \frac{\alpha}{2\pi} u(\xi) + \oint_{\Gamma} u(x) \frac{(x_i - \xi_i) n_i}{2\pi |x_i - \xi_i|^2} d\Gamma - \int_{\Gamma} q(x) \frac{\ln(x_i - \xi_i)}{2\pi} d\Gamma ,$$

(1.3.86)

or

$$\underbrace{\left(1 - \frac{\alpha}{2\pi}\right)}_{c(\xi)} u(\xi) + \oint_{\Gamma} q^*(x,\xi) u(x) d\Gamma = \int_{\Gamma} u^*(x,\xi) q(x) d\Gamma .$$

(1.3.87)

The factor $c(\xi)$ is the free term coefficient and can be interpreted as the fraction of $u(\xi)$ that lies inside Ω:

$$c(\xi) = \begin{cases} 1 - \frac{\alpha}{2\pi} & \text{for} \quad \xi \in \Gamma \\ 1 & \text{for} \quad \xi \in \Omega \\ 0 & \text{for } \xi \notin \Gamma, \xi \notin \Omega \end{cases} .$$

(1.3.88)

The boundary is divided into boundary elements, and the boundary variables are interpolated by piecewise continuous functions, e.g., polynomials, so that an approximate calculation of the boundary integrals becomes possible. This approach is called discretisation and will be described in the following.

1.3.2 Discretisation of the Boundary

To approximate the geometry, the boundary Γ is divided into E boundary elements $\Gamma^{(1)},\ldots,\Gamma^{(E)}$, each of which possesses one or more nodes.

Inside the element (e) with local coordinate s, the potential $u^{(e)}$ and flux $q^{(e)}$ are interpolated using shape function $\Psi_m(s)$ and nodal $\breve{u}_m^{(e)}$ and $\breve{q}_m^{(e)}$ as follows:

$$u^{(e)}(s) = \sum_{m=1}^{M^{(e)}} \Psi_m(s) \breve{u}_m^{(e)} \quad \text{and} \quad q^{(e)}(s) = \sum_{m=1}^{M^{(e)}} \Psi_m(s) \breve{q}_m^{(e)} ,$$

(1.3.89)

or in matrix notation

$$u^{(e)}(s) = \Psi^T(s) \breve{u}^{(e)} \qquad \text{and} \qquad q^{(e)}(s) = \Psi^T(s) \breve{q}^{(e)} ,$$

(1.3.90)

where $\breve{u}_m^{(e)}$, $\breve{q}_m^{(e)}$, Ψ are $(M \times 1)$-matrices. In 2-D analysis, the simplest shape functions are constant and linear shape functions.

(I) Constant Shape Functions Elements with constant shape functions possess only one located in the middle of the element, as shown in figure 1.3. The values of $u^{(e)}$ and $q^{(e)}$ are constant throughout the element and correspond to the value at the node. This means that $\Psi_1(s) = 1$ and

$$u^{(e)}(s) = \Psi_1(s)\breve{u}_1^{(e)} = \breve{u}_1^{(e)} \quad \text{and} \quad q^{(e)}(s) = \Psi_1(s)\breve{q}_1^{(e)} = \breve{q}_1^{(e)}. \tag{1.3.91}$$

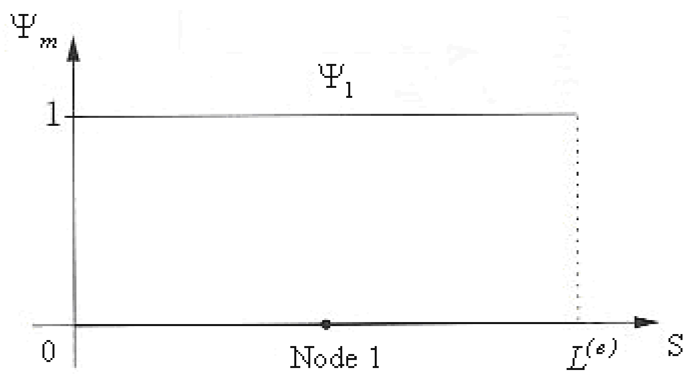


Figure 1.3. Constant shape function

(II) Linear Shape Functions Using two nodes per element, we can interpolate the potential $u^{(e)}(s)$ linearly over the length $L^{(e)}$ of the element as shown in Figure 1.4. With the nodal values $\breve{u}_1^{(e)}$ and $\breve{u}_2^{(e)}$, we obtain

$$\begin{aligned} u^{(e)}(s) &= \breve{u}_1^{(e)} + \frac{\breve{u}_2^{(e)} - \breve{u}_1^{(e)}}{L^{(e)}} s \\ &= \underbrace{\left(1 - \frac{s}{L^{(e)}}\right)}_{\Psi_1} \breve{u}_1^{(e)} + \underbrace{\frac{s}{L^{(e)}}}_{\Psi_2} \breve{u}_2^{(e)}, \end{aligned} \tag{1.3.92}$$

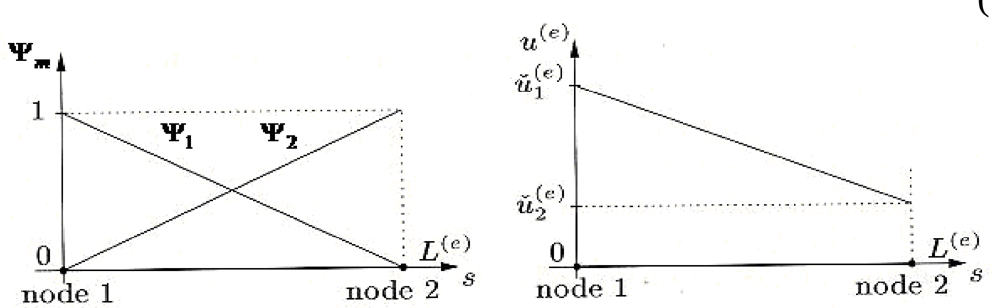


Figure 1.4. Linear interpolation displacements within the element $\Gamma^{(e)}$

or in matrix notation

$$u^{(e)}(s) = \begin{bmatrix} \Psi_1 & \Psi_2 \end{bmatrix} \begin{bmatrix} \breve{u}_1^{(e)} \\ \breve{u}_2^{(e)} \end{bmatrix} = \Psi^T \breve{u}^{(e)}. \tag{1.3.93}$$

Now, using the discretisation (1.3.89), we obtain for (1.3.87) in 2-D

$$c(\xi)u(\xi)+\sum_{e=1}^{E}\int_{\Gamma^{(e)}}\left(\sum_{m=1}^{M}\Psi_m u_m^{(e)}\right)q^* d\Gamma=\sum_{e=1}^{E}\int_{\Gamma^{(e)}}\left(\sum_{m=1}^{M}\Psi_m q_m^{(e)}\right)u^* d\Gamma\ . \tag{1.3.94}$$

Since the nodal values $\breve{u}_m^{(e)}$ and $\breve{q}_m^{(e)}$ are constant, we can write

$$c(\xi)u(\xi)+\sum_{e=1}^{E}\left(\sum_{m=1}^{M}\breve{u}_m^{(e)}\int_{\Gamma^{(e)}}\Psi_m q^* d\Gamma\right)=\sum_{e=1}^{E}\left(\sum_{m=1}^{M}\breve{q}_m^{(e)}\int_{\Gamma^{(e)}}\Psi_m u^* d\Gamma\right), \tag{1.3.95}$$

which is the discretised from the boundary integral equation.

If constant elements are used, the node is usually located at the center of the element. In this case, $\alpha=\pi$, and we obtain

$$c(\xi)=1-\frac{\alpha}{2\pi}=\frac{1}{2}. \tag{1.3.96}$$

If the load point in the field point are located on the same element, the vector $r_i=(x_i-\xi_i)$ is perpendicular to n_i and therefore

$$(x_i-\xi_i)n_i=0\ . \tag{1.3.97}$$

1.3.3 The Collocation Method

The discretised boundary integral equation (1.3.95) is now used to set up a system of equations for the determination of the unknown boundary values.

While this can be done in a number of ways using the method of weighted residuals, we will only describe the collocation method, which is by far the most frequently used approach due to its versatility and computational efficiency. An alternative approach is the symmetric Galerkin BEM (Brebbia [18]), which involves at double surface integration. For medium to large-scale problems, this additional numerical cost can often be compensated by the advantages resulting from the symmetry of the equations.

In the collocation method, we place the load point ξ sequentially on all nodesof the discretisation. This way, the free term $c(\xi)u(\xi)$ contains the potential at the discretisation node so that no additional unknown is introduced. When using linear and higher-order polynomial shape functions, some of the nodes belong to more than one element, so it is advantageous to introduce a global node numbering $(n=1,\ldots,N)$ that is independent of the elements.

By placing the load point on the first global node, we obtain the equation

$$\underbrace{\breve{u}_1\left(\int_{\Gamma^{(1,e)}} \Psi_1 q^*(x,\xi^1)d\Gamma + c_1\right)}_{\hat{H}_{11}} + \cdots + \breve{u}_N \underbrace{\int_{\Gamma^{(N,e)}} \Psi_N q^*(x,\xi^1)d\Gamma}_{H_{1N}}$$

$$= \breve{q}_1 \underbrace{\int_{\Gamma^{(1,e)}} \Psi_1 u^*(x,\xi^1)d\Gamma}_{G_{11}} + \cdots + \breve{q}_N \underbrace{\int_{\Gamma^{(N,e)}} \Psi_N u^*(x,\xi^1)d\Gamma}_{G_{1N}} \quad .$$

(1.3.98)

In (1.3.98), the integral $\int_{\Gamma^{(n,e)}} (\cdot) d\Gamma$ denotes the sum of all integrals over the elements (e) on which the node with the global number n is located, and Ψ_n is the corresponding shape function. In matrix notation, we can write (1.3.98) as follows:

$$\begin{bmatrix} \hat{H}_{11} & H_{12} \cdots & H_{1N} \end{bmatrix} \begin{bmatrix} \breve{u}_1 \\ \breve{u}_2 \\ \vdots \\ \breve{u}_N \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \cdots & G_{1N} \end{bmatrix} \begin{bmatrix} \breve{q}_1 \\ \breve{q}_2 \\ \vdots \\ \breve{q}_N \end{bmatrix} ,$$

(1.3.99)

where $\hat{H}_{11}$ denotes the entry that contains the free term coefficient $c_1 = c(\xi^1)$.

By collocating the load point on the nodes 2 to N, we obtain the missing equations, which we assemble to form the system

$$\begin{bmatrix} \hat{H}_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{bmatrix} \begin{bmatrix} \breve{u}_1 \\ \breve{u}_2 \\ \vdots \\ \breve{u}_N \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} \breve{q}_1 \\ \breve{q}_2 \\ \vdots \\ \breve{q}_N \end{bmatrix} ,$$

(1.3.100)

or in matrix notation

$$H\breve{u} = G\breve{q} \quad .$$

(1.3.101)

The diagonal eleme nts of the matrices H and G contain the strongly and weakly singular integrals, respectively, because $r = |x_i - \xi_i|$ vanishes when the node ξ lies on the element over which the integration is carried out. All other matrix elements contain regular integrals. Since both nodal vectors $\breve{u}$ and $\breve{q}$ in (1.3.101) contain known as well as unknown boundary values, we have to rewrite the equations with all unknowns appearing in a vector $\breve{y}$ on one side,

$$A\breve{y} = f \quad ,$$

(1.3.102)

Where the known boundary values are multiplied with the corresponding matrix entries to yield the vector f. The system (1.3.102) can now be solved with standard direct or iterative methods.

By means of a simple example, we will now illustrate how to the rearrange the equations in (1.3.101). Consider the 2-D region shown in figure 1.5. At the

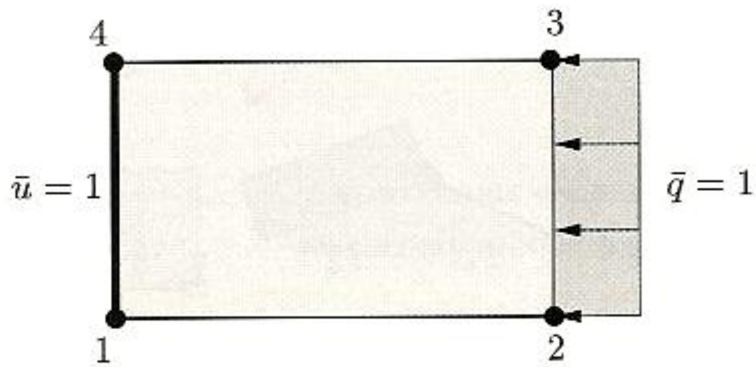


Figure 1.5. Rectangular region with prescribed potential $\bar{u} = 1$ on one face and prescribed flux $\bar{q} = 1$ on the opposite face

boundary nodes 1 and 4, the potential $u = \bar{u}$ is prescribed and the flux q is unknown, while at boundary nodes 2 and 3, the flux $q = \bar{q}$ is prescribed and the potential u is unknown. The system of equations obtained with the Boundary Element Method is the given by.

$$\begin{bmatrix} \hat{H}_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} / \\ ? \\ ? \\ / \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} ? \\ / \\ / \\ ? \end{bmatrix}, \tag{1.3.103}$$

where '/' denotes prescribed and '?' denotes unknown boundary values. Rearranging the system by separating known and unknown boundary values yields

$$\underbrace{\begin{bmatrix} -G_{11} & H_{12} & H_{13} & -G_{14} \\ -G_{21} & H_{22} & H_{23} & -G_{24} \\ -G_{31} & H_{32} & H_{33} & -G_{34} \\ -G_{41} & H_{42} & H_{43} & -G_{44} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}}_{\breve{y}} = \underbrace{\begin{bmatrix} -\hat{H}_{11} & G_{12} & G_{13} & -H_{14} \\ -H_{21} & G_{22} & G_{23} & -H_{24} \\ -H_{31} & G_{32} & G_{33} & -H_{34} \\ -H_{41} & G_{42} & G_{43} & -H_{44} \end{bmatrix} \begin{bmatrix} / \\ / \\ / \\ / \end{bmatrix}}_{f}, \tag{1.3.104}$$

which corresponds to (1.3.102).

CHAPTER (II)

(Boundary element solution of steady-state temperature distribution in homogeneous media)

2.1 Introduction

The boundary element method (BEM) is an integral-equation-based mathematically was numrical technique that offers many advantages over FDM, FVM, or FEM. Theoretical development of the BEM relies on the formulation of the boundary integral equation that is predicated on the availability of the so-called Green's free-space solution for the problem of interest. Theoretical background and numerical implementation of the BEM can be found in the monographs by Brebbia and Walker [22], Brebbia et.al. [21], Gipson [35], and Banerjee [16], and in the book by Brebbia and Dominguez [20]. One of the most striking features of BEM is that, of many field problems of engineering, a boundary integral equation is discretized to solve the field problem of interest. Consequently, only the bounding surface of the domain is discretized, thereby reducing the dimension of the problem by one. For instance, in the analysis of linear and non-linear isotropic steady-state heat conduction and in linear elasticity, a boundary discretization is only required to resolve the temperature or stress field. Thus, for a certain class of problems, for which Green's free-space solutions are available, the BEM solution can be expressed in the terms of boundary integrals only.

2.2 formulation of the problem

The two - dimensional steady-state temperature distribution may be written in non- dimensional form as follows

$$\frac{\partial^2 \mathrm{T}}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{2.2.1}$$

Equation (2.2.1) is to be solved in a two – dimensional region R bounded by a simple closed curve D subject to the boundary condition

$$\mathrm{T} = g(x, y) \qquad \text{for} \quad (x, y) \in D_1$$
$$\frac{\partial \mathrm{T}}{\partial n} = h(x, y) \qquad \text{for } (x, y) \in D_2$$

(2.2.2)

where g and h are suitably prescribed functions and D_1 and D_2 are intersecting curves such that $D_1 \cup D_2 = D$ Refer to Figure 2.1 for a geometrical sketch of the problem.

The normal derivative $\partial \mathrm{T} / \partial n$ in Eq. 2.2.1 is defined by

$$\frac{\partial \mathrm{T}}{\partial n} = n_x \frac{\partial \mathrm{T}}{\partial x} + n_y \frac{\partial \mathrm{T}}{\partial y}$$

(2.2.3)

where n_x and n_y are respectively the x and y components of a unit normal vector to the curve D. Here the unit normal vector $[n_x, n_y]$ on D is taken to be pointing away from the region R. Note that the normal vector may vary from point to point on D. Thus, $[n_x, n_y]$is a function of x and y

The boundary conditions given in Eq. (2.2.2) are assumed to be properly posed so that the boundary value problem has a unique solution, that is, it is assumed that one can always find a function $\mathrm{T}(x, y)$ satisfying Eqs. (2.2.1) , (2.2.2) and that there is only one such function.

For a particular example of practical situations involving the boundary value problem above, one may mention the classical heat conduction problem where T denotes the steady-state temperature in an isotropic solid. Eq. (2.2.1) is then the temperature governing equation derived, under certain assumptions, from the law of conservation of energy together with the Fourier's heat flux model. The heat flux out of the region R across the boundary D is given by $-k \partial \mathrm{T} / \partial n$, where k is the thermal heat conductivity of the solid. Thus, the boundary conditions in Eq. (2.2.2) imply that at each and every given point on D either the temperature or the heat flux (but not both) is known. To determine the temperature field in the solid, one has to solve Eq. (2.2.1) in R to find the solution that satisfies the prescribed boundary conditions on D.

In general, it is difficult (if not impossible) to solve exactly the boundary value problem defined by Eqs. (2.2.1) , (2.2.2). The mathematical complexity involved depends on the geometrical shape of the region R and the boundary conditions given in Eq. (2.2.2) Exact solution can only be found for relatively simple geometries of R (such as square region) together with particular boundary conditions for more complicated geometries or general boundary conditions, one may have to resort to approximate techniques for solving Eqs. (2.2.1) , (2.2.2).

We show how in this chapter a boundary integral solution can be derived for Eq. (2.2.1) and apply example to obtain a simple boundary element procedure for approximately solving the boundary value problem under consideration.

$$\Psi(x,y,\xi,\eta)=\frac{1}{4\pi}\ln[(x-\xi)^2+(y-\eta)^2]. \tag{2.2.4}$$

We refer $\Psi(x,y,\xi,\eta)$ in Eq. (2.2.4) as the fundamental solution of two-dimensional Laplac's equation may be written in the following form (Ang [16]). Note that $\Psi(x,y,\xi,\eta)$ satisfies Eq. (2.2.1) every where except at (ξ,η) where it is not well defined.

If T_1 and T_2 are any two solutions ot Eq. (2.2.1) in the region R bounded by the simple closed curve D then it can be shown that

$$\int_D (T_2\frac{\partial T_1}{\partial n}-T_1\frac{\partial T_2}{\partial n})ds(x,y)=0 \tag{2.2.5}$$

Since T_1 and T_2 are solutions of Eq. (2.2.1) , we may write

$$\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial y^2}=0,$$

$$\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial y^2}=0.$$

If we multiply the first equation by T_2 and the second one by T_1 and take the difference of the resulting equations , we obtain

$$\frac{\partial}{\partial x}(T_2\frac{\partial T_1}{\partial x}-T_1\frac{\partial T_2}{\partial x})+\frac{\partial}{\partial y}(T_2\frac{\partial T_1}{\partial y}-T_1\frac{\partial T_2}{\partial y})=0$$

which can be integrated over R to give

$$\iint_R [\frac{\partial}{\partial x}(T_2\frac{\partial T_1}{\partial x}-T_1\frac{\partial T_2}{\partial x})+\frac{\partial}{\partial y}(T_2\frac{\partial T_1}{\partial y}-T_1\frac{\partial T_2}{\partial y})]dxdy=0$$

Application of the divergence theorem to convert the double integral over R into a line integral over D yields

$$\int_D [(T_2\frac{\partial T_1}{\partial x}-T_1\frac{\partial T_2}{\partial x})n_x+(T_2\frac{\partial T_1}{\partial y}-T_1\frac{\partial T_2}{\partial y})n_y]ds(x,y)=0$$

which is essentially Eq. (2.2.5).

Together with the fundamental solution given by Eq. (2.2.4) reciprocal relation in Eq. (2.2.5) can be used to derive a useful boundary integral solution for the two-dimensional Laplace's equation.

2.3 Boundary Element Procedure

Using the same technique as in sections 1.1 and 1.3 (see chapter I)

Let us take $T_1 = \Psi(x, y; \xi, \eta)$ (the fundamental solution as defined in Eq. (2.2.4)) and $T_2 = T$, where T is the required solution of the interior boundary value problem defined by Eqs. (2.2.1), (2.2.2).

Since $\Psi(x, y; \xi, \eta)$ is not well defined at the point (ξ, η), the reciprocal relation in Eq. (2.2.5)is valid for $T_1 = \Psi(x, y; \xi, \eta)$ and $T_2 = T$ only if (ξ, η) does not lie in the region $R \bigcup D$. Thus, (Ang [13])

$$\int_D [\mathrm{T}(x, y)\frac{\partial}{\partial n}(\Psi(x, y; \xi, \eta)) - \Psi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\mathrm{T}(x, y))]ds(x, y) = 0$$

$$\text{for } (\xi, \eta) \notin R \bigcup D. \tag{2.3.6}$$

A more interesting and useful integral equation than Eq. (2.3.6) can be derived from Eq. (2.2.5) if we take the point (ξ, η) to lie in the region $R \bigcup D$.

For the case in which (ξ, η) lies in the interior of R, Eq. (2.2.5) is valid if we replace D by $D \bigcup D_\varepsilon$, where D_ε is a circle of center (ξ, η) and radius ε as shown in Figure (2.2). This is because $\Psi(x, y; \xi, \eta)$ and its first order partial derivatives (with respect to x or y) are well defined in the region between D and D_ε. Thus, for D and D_ε in Figure (2.2) we can write (Gaul, et al. [34])

$$\int_{D \bigcup D_\varepsilon} [\ T(x, y)\frac{\partial}{\partial n}(\Psi(x, y; \xi, \eta)) - \Psi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\mathrm{T}(x, y))]ds(x, y) = 0$$

that is,

$$\int_D [T(x, y)\frac{\partial}{\partial n}(\Psi(x, y; \xi, \eta)) - \Psi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\mathrm{T}(x, y))]ds(x, y)$$

$$= -\int_{D_\varepsilon} [T(x, y)\frac{\partial}{\partial n}(\Psi(x, y; \xi, \eta)) - \Psi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\mathrm{T}(x, y))]ds(x, y). \tag{2.3.7}$$

Eq. (2.3.7) holds for any radius $\varepsilon > 0$, so long as the circle D_ε (in Figure 2.2) lies completely inside the region bounded by D. Thus, we may let $\varepsilon \to 0^+$ in Eq. (2.3.7). This gives

$$\int_{D}[T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta))-\Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\mathrm{T}(x,y))]ds(x,y)$$

$$=-\lim_{\varepsilon\to 0^{+}}\int_{D_\varepsilon}[T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta))-\Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\mathrm{T}(x,y))]ds(x,y).$$

(2.3.8)

Using polar coordinates r and θ centered about (ξ,η) as defined by $x-\xi=r\cos\theta$ and $y-\eta=r\sin\theta$, we may write

$$\Psi(x,y;\xi,\eta)=\frac{1}{2\pi}\ln(r),$$

$$\frac{\partial}{\partial n}[\Psi(x,y;\xi,\eta)]=n_x\frac{\partial}{\partial x}[\Psi(x,y;\xi,\eta)]+n_y\frac{\partial}{\partial y}[\Psi(x,y;\xi,\eta)]$$

$$=\frac{n_x\cos\theta+n_y\sin\theta}{2\pi r}\ .$$

(2.3.9)

The Taylor's series of $T(x,y)$ about the point (ξ,η) is given by

$$T(x,y)=\sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m T}{\partial x^k\partial y^{m-k}})\Big|_{(x,y)=(\xi,\eta)}\frac{(x-\xi)^k(y-\eta)^{m-k}}{k!(m-k)!}.$$

On the circle D_ε, $r=\varepsilon$. Thus,

$$T(x,y)=\sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m}{\partial x^k\partial y^{m-k}}[T(x,y)])\Big|_{(x,y)=(\xi,\eta)}\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!}$$

$$\text{for } (x,y)\in D_\varepsilon$$

(2.3.10)

Similarly, we may write

$$\frac{\partial}{\partial n}[T(x,y)]=\sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m}{\partial x^k\partial y^{m-k}}\{\frac{\partial}{\partial n}[T(x,y)]\})\Big|_{(x,y)=(\xi,\eta)}$$

$$\times\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!}\quad\text{for } (x,y)\in D_\varepsilon.$$

(2.3.11)

Using Eqs. (2.3.9), (2.3.10) and (2.3.11) and writing $ds(x,y)=\varepsilon d\theta$ with θ ranging from 0 to 2π, we may now attempt to evaluate the limit on the write hand side of Eq. (2.3.8). On D_ε, the normal vector $[n_x,n_y]$ is given by $[-\cos\theta,-\sin\theta]$. Thus,

$$\int_{D_\varepsilon} T(x,y)\frac{\partial}{\partial n}[\Psi(x,y;\xi,\eta)]ds(x,y)$$
$$=-\frac{1}{2\pi}T(\xi,\eta)\int_0^{2\pi}d\theta$$
$$-\frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\varepsilon^m}{k!(m-k)!}(\frac{\partial^m T}{\partial x^k \partial y^{m-k}})\Big|_{(x,y)=(\xi,\eta)}\int_0^{2\pi}\cos^k\theta\sin^{m-k}\theta d\theta$$
$$\to -T(\xi,\eta) \text{ as } \varepsilon\to 0^+,$$

(2.3.12)

and

$$\int_{D_\varepsilon} \Psi(x,y)\frac{\partial}{\partial n}[T(x,y;\xi,\eta)]ds(x,y)$$
$$=\frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m}{\partial x^k \partial y^{m-k}}(\frac{\partial}{\partial n}[T(x,y)]))\Big|_{(x,y)=(\xi,\eta)}$$
$$\times\frac{\varepsilon^{m+1}\ln(\varepsilon)}{k!(m-k)!}\int_0^{2\pi}\cos^k\theta\sin^{m-k}\theta d\theta$$
$$\to 0 \text{ as } \varepsilon\to 0^+,$$

2.3.13)

since $\varepsilon^{m+1}\ln(\varepsilon)\to 0$ as $\varepsilon\to 0^+$ for $m=0,1,2,\ldots$

Consequently, as $\varepsilon\to 0^+$, Eq. (2.3.8) yields

$$T(\xi,\eta)=\int_D [T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta))$$
$$-\Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y))]ds(x,y)$$
$$\text{for} \quad (\xi,\eta)\in R.$$

(2.3.14)

Together with Eq. (2.2.4), Eq.(2.3.14) provides us with a boundary integral solution for the two-dimensional Laplace's equation. If both T and $\partial T/\partial n$ are known at all point on D, the line integral in Eq. (2.3.14) can be evaluated (at least in theory) to calculate T at any point (ξ,η) in the interior of R. From the boundary conditions (2.2.2), at any given point on D, either T or $\partial T/\partial n$, not both, is unknown, however.

To solve the interior boundary value problem, we must find the unknown T and $\partial T/\partial n$ on D_2 and D_1 respectively. As we shall see later on, this may be done through manipulation of data on boundary D only, if we can derive a boundary integral formula for (ξ,η), similar to the one in Eq. (2.3.14), for general point (ξ,η) that lies on D.

For the case in which the point (ξ,η) lies on D, Eq. (2.2.5) holds if we replace the curve D by $C \cup C_\varepsilon$, where the curves C and C_ε are as shown in Figure (2.3). (If D_ε is the circle of center (ξ,η) and radius ε, then C, is the part of D that lies outside D_ε and C_ε is the part of D_ε that is inside R.) Thus,

$$\int_C [T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)) - \Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y))]ds(x,y)$$
$$= -\int_{C_\varepsilon} [T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)) - \Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y))]ds(x,y). \tag{2.3.15}$$

Let us examine what happens to Eq. (2.3.15) when we let $\varepsilon \to 0^+$.

As $\varepsilon \to 0^+$, the curve C tends to D. Thus we may write

$$\int_D [T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)) - \Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y))]ds(x,y)$$
$$= -\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon} [T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta))$$
$$-\Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y))]ds(x,y). \tag{2.3.16}$$

Note that, unlike in Eq. (2.3.8), the line integral over D in Eq. (2.3.16) is improper as its integrand is not well defined at (ξ,η) which lies on D. Strictly speaking, the line integration should be over the curve D without an infinitesimal segment that contains the point (ξ,η), that is, line integral over D in Eq. (2.3.16) has to be interpreted in the Cauchy principal sense if (ξ,η) lies on D.

To evaluate the limit on the right hand side of Eq. (2.3.16), we need to know what happens to C_ε when we let $\varepsilon \to 0^+$. Now if (ξ,η) lies on a smooth part of D (not at where the gradient of the curve changes abruptly, that is not at a corner point, if there is any), one can intuitively see that the part of D inside D_ε approaches an infinitesimal straight line as $\varepsilon \to 0^+$. Thus, we expect C_ε to tend to a semi-circle as $\varepsilon \to 0^+$, if (ξ,η) lies on smooth part of D. It follows that in attempting to evaluate the limit on the right hand side of Eq. (2.3.16) we have to integrate over only half a circle (instead of a full circle as in the case of Eq. (2.3.8)).

Modifying Eqs. (2.3.12) and (2.3.13), we obtain

$$\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon}[T(x,y)\frac{\partial}{\partial n}[\Psi(x,y;\xi,\eta)]ds(x,y) = -\frac{1}{2}T(\xi,\eta),$$

$$\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon}\Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}[T(x,y)]ds(x,y) = 0.$$

Hence Eq. (2.3.11) gives

$$\frac{1}{2}T(\xi,\eta) = \int_D [T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)) - \Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y))]ds(x,y)$$

for (ξ,η) lying on smooth part of D.

(2.3.17)

Together with the boundary conditions in Eq. (2.2.2) , Eq. (2.3.17) may applied to obtain a numerical procedure for determining the unknown T and /or $\partial T/\partial n$ on the boundary D. Once T and $\partial T/\partial n$ are known at all points on D, the solution of the interior boundary value problem defined by Eqs. (2.2.1), (2.3.14) and (2.3.17) as a single equation given by

$$\lambda(\xi,\eta)T(\xi,\eta) = \int_D [T(x,y)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)) - \Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y))]ds(x,y)$$

(2.3.18)

if we define

$$\lambda(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta)\notin R\cup D \\ 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of D.} \\ 1 & \text{if } (\xi,\eta)\in R \end{cases}$$

(2.3.19)

We now show how Eq. (2.3.18) may be applied to obtain a simple boundary element procedure with constant elements for solving numerically the interior boundary problem defined by Eqs. (2.2.1),(2.2.2).

The boundary D is approximated as an N-sided polygon with sides $D^{(1)}, D^{(2)},\ldots,D^{(N-1)}$ and $D^{(N)}$, that is,

$$D \cong D^{(1)}\cup D^{(2)}\cup\ldots\cup D^{(N-1)}\cup D^{(N)}.$$

(2.3.20)

The sides or the boundary elements $D^{(1)}, D^{(2)},\ldots,D^{(N-1)}$ and $D^{(N)}$, are constructed as follows. We put N well spaced out $(x^{(1)},y^{(1)})$, $(x^{(2)},y^{(2)})$ …, $(x^{(N-1)},y^{(N-1)})$ and $(x^{(N)},y^{(N)})$ on D, in the order given, following the counter clockwise direction. Defining $(x^{N+1}, y^{N+1}) = (x^{(1)}, y^{(1)})$, we take $D^{(k)}$ to be the boundary element from $(x^{(k)}, y^{(k)})$ to $(x^{(k+1)}, y^{(k+1)})$ for $k = 1,2,\ldots,N$.

For a simple approximation of T and $\partial \mathrm{T}/\partial n$ on the boundary D, we assume that these functions are constants over each of the boundary elements. Specifically, we make the approximation:

$$T \cong \overline{T}^{(k)} \text{ and } \frac{\partial \mathrm{T}}{\partial n} \cong \overline{p}^{(k)} \text{ for } (x, y) \in D^{(k)} (k = 1,2,\ldots,N) \tag{2.3.21}$$

where $\overline{T}^{(k)}$ and $\overline{p}^{(k)}$ are respectively the values of T and $\partial T / \partial n$ at the midpoint $D^{(k)}$.

With Eqs. (2.3.20) and (2.3.21), we find that Eq. (2.3.18) can be approximately written as

$$\lambda(\xi,\eta)T(\xi,\eta) = \sum_{k=1}^{N} \{\overline{T}^{(k)} F_2^{(k)}(\xi,\eta) - \overline{P}^{(k)} F_1^{(k)}(\xi,\eta)\}, \tag{2.3.22}$$

where

$$F_1^{(k)}(\xi,\eta) = \int_{D^{(k)}} \Psi(x, y;\xi,\eta) ds(x, y),$$

$$F_2^{(k)}(\xi,\eta) = \int_{D^{(k)}} \frac{\partial}{\partial n}[\Psi(x, y;\xi,\eta)] ds(x, y). \tag{2.3.23}$$

For a given k, either $\overline{T}^{(k)}$ or $\overline{p}^{(k)}$ (not both) is known from the boundary conditions in Eq.(2.2.2). Thus, there are N unknown constants on the right hand side of Eq. (2.3.22). To determine their values, we have to generate N equations containing the unknowns.

If we let (ξ,η) in Eq. (2.3.22) be given in turn by the midpoints of $D^{(1)}, D^{(2)},\ldots,D^{(N-1)}$ and $D^{(N)}$ we obtain

$$\frac{1}{2}\overline{T}^{(m)} = \sum_{k=1}^{N} \{\overline{T}^{(k)} F_2^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) - \overline{p}^{(k)} F_1^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)})\} \quad \text{for } m = 1,2,\ldots,N, \tag{2.3.24}$$

where $(\overline{x}^{(m)}, \overline{y}^{(m)})$ is the midpoint of $D^{(m)}$.

In the derivation of Eq. (2.3.24), we take $\lambda(\overline{x}^{(m)}, \overline{y}^{(m)}) = 1/2$, since $(\overline{x}^{(m)}, \overline{y}^{(m)})$ being the midpoint of $D^{(m)}$ lies on a smooth part of the approximate boundary $D^{(1)} \cup D^{(2)} \cup \ldots \cup D^{(N-1)} \cup D^{(N)}$.

Eq. (2.3.24) constitutes a system of N linear algebraic equations containing the N unknown on the right hand side of Eq. (2.3.22). We may rewrite it as

$$\sum_{k=1}^{N} a^{(mk)} z^{(k)} = \sum_{k=1}^{N} b^{(mk)} \qquad \text{for } m = 1,2,\ldots,N,$$

(2.3.25)

where $a^{(mk)}, b^{(mk)}$ and $z^{(k)}$ are defined by

$$a^{(mk)} = \begin{cases} -F_1^{(k)}(\bar{x}^{(m)}, \bar{y}^{(m)}) & \text{if } T \text{ is specified over } D^{(k)}, \\ F_2^{(k)}(\bar{x}^{(m)}, \bar{y}^{(m)}) - \frac{1}{2}\delta^{(m,k)} & \text{if } \partial T/\partial n \text{ is specified over } D^{(k)}, \end{cases}$$

$$b^{(mk)} = \begin{cases} \bar{T}^{(k)}(-F_2^{(k)}(\bar{x}^{(m)}, \bar{y}^{(m)}) + \frac{1}{2}\delta^{(mk)}) & \text{if } T \text{ is specified over } D^{(k)}, \\ \bar{p}^{(k)} F_1^{(k)}(\bar{x}^{(m)}, \bar{y}^{(m)}) & \text{if } \partial T/\partial n \text{ is specified over } D^{(k)}, \end{cases}$$

$$\delta^{(mk)} = \begin{cases} 0 & \text{if } m \neq k \\ 1 & \text{if } m = k \end{cases}$$

$$z^{(k)} = \begin{cases} \bar{p}^{(k)} & \text{if } T \text{ is specified over } D^{(k)} \\ \bar{T}^{(k)} & \text{if } \partial T/\partial n \text{ is specified over } D^{(k)} \end{cases}$$

(2.3.26)

Note that $z^{(1)}, z^{(2)}, \ldots, z^{(N-1)}$ and z^{N} are the N unknown constants on the right hand side of Eq. (2.3.22), while $a^{(mk)}$ and $b^{(mk)}$ are known coefficients.

Once Eq. (2.3.25) is solved for the unknowns $z^{(1)}, z^{(2)}, \ldots, z^{(N-1)}$ and z^{N}, the values of T and $\partial T/\partial n$ over the element $D^{(k)}$, as given by $\bar{T}^{(k)}$ and $\bar{p}^{(k)}$ respectively, are known for $k = 1,2,\ldots,N$. Eq. (2.3.22) with $\lambda(\xi,\eta) = 1$ then providesus with an explicit formula for computing T in the interior of R, that is,

$$T(\xi,\eta) \cong \sum_{k=1}^{N} \{\bar{T}^{(k)} F_2^{(k)}(\xi,\eta) - \bar{p}^{(k)} F_1^{(k)}(\xi,\eta)\} \quad \text{for } (\xi,\eta) \in R.$$

(2.3.27)

To summarize a boundary element solution of the interior boundary value problem defined by Eqs.(2.2.1),(2.2.2) is given by Eq.(2.3.27) together with Eqn. (2.3.20) and (2.3.21), the solution is said to be obtained using constant elements. Analytical formula for calculating $F_1^{(k)}(\xi,\eta)$ and $F_2^{(k)}(\xi,\eta)$ in Eq. (2.3.23) are given in Eqs. (2.3.32), (2.3.33), (2.3.35) and (2.3.36) (together with Eq. (2.3.30)) below.

The boundary element solution above requires the evaluation of $F_1^{(k)}(\xi,\eta)$ and $F_2^{(k)}(\xi,\eta)$. These functions are defined in terms of line integrals over $D^{(k)}$ as given in Eq. (2.3.23).
The line integrals can be worked out analytical as follows.

Points on the element $D^{(k)}$ may be described using the parametric equations

$$\left.\begin{aligned} x &= x^{(k)} - tl^{(k)}n_y^{(k)} \\ y &= y^{(k)} - tl^{(k)}n_x^{(k)} \end{aligned}\right\} \text{ from } t=0 \text{ to } t=1, \tag{2.3.28}$$

where $l^{(k)}$ is the length of $D^{(k)}$ and $[n_x^{(k)}, n_y^{(k)}] = [y^{(k+1)} - y^{(k)}, x^{(k)} - x^{(k+1)}]/l^{(k)}$ is the unit normal vector to $D^{(k)}$ pointing away from R.

For $(x,y) \in D^{(k)}$, we find that $ds(x,y) = \sqrt{(dx)^2 + (dy)^2} = l^{(k)}dt$ and

$$(x-\xi)^2 + (y-\eta)^2 = A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta), \tag{2.3.29}$$

where

$$\begin{aligned} A^{(k)} &= [l^{(k)}]^2, \\ B^{(k)}(\xi,\eta) &= [-n_y^{(k)}(x^{(k)} - \xi) + (y^{(k)} - \eta)n_x^{(k)}](2l^{(k)}), \\ E^{(k)}(\xi,\eta) &= (x^{(k)} - \xi)^2 + (y^{(k)} - \eta)^2. \end{aligned} \tag{2.3.30}$$

The parameters in Eq. (2.3.30) satisfy $4A^{(k)}E^{(k)}(\xi,\eta) - [B^{(k)}(\xi,\eta)]^2 \geq 0$ for any point (ξ,η). To see why this is true, consider the straight line defined by the parametric equations $x = x^{(k)} - tl^{(k)}n_y^{(k)}$ and $y = y^{(k)} - tl^{(k)}n_x^{(k)}$ for $-\infty < t < \infty$. Note that $D^{(k)}$ is a subset of this straight line (given by the parametric equations from $t=0$ to $t=1$). Eq. (2.3.29) also holds for any point (x,y) lying on the extended line. If (ξ,η) dose not lie on the line then $A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta) > 0$ for all real values of t (that is, for all points (x,y) on the line) and hence $4A^{(k)}E^{(k)}(\xi,\eta) - [B^{(k)}(\xi,\eta)]^2 > 0$. On the other hand, if (ξ,η) is on the line, we can find exactly one point (x,y) such that $A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta) = 0$. As each point (x,y) on the line given by unique value of t, we conclude that $4A^{(k)}E^{(k)}(\xi,\eta) - [B^{(k)}(\xi,\eta)]^2 = 0$ for (ξ,η) lying on the line.

From Eqs. (2.3.23), (2.3.28) and (2.3.29), $F_1^{(k)}(\xi,\eta)$ and $F_2^{(k)}(\xi,\eta)$ may be written as

$$F_1^{(k)}(\xi,\eta) = \frac{l^{(k)}}{4\pi}\int_0^1 \ln[A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta)]dt,$$

$$F_2^{(k)}(\xi,\eta) = \frac{l^{(k)}}{2\pi}\int_0^1 \frac{n_x^{(k)}(x^{(k)}-\xi)+n_y^{(k)}(y^{(k)}-\eta)}{A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta)}dt. \tag{2.3.31}$$

The second integral in Eq. (2.3.31) is the easiest one to work out for the case in which $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2 = 0$. For this case, the point (ξ,η) lies on the straight line of which the element $D^{(k)}$ is a subset. Thus, the vector $[x^{(k)}-\xi, y^{(k)}-\eta]$ is perpendicular to $[n_x^{(k)}, n_y^{(k)}]$, that is

$n_x^{(k)}(x^{(k)}-\xi)+n_y^{(k)}(y^{(k)}-\eta) = 0$, and we obtain

$$F_2^{(k)}(\xi,\eta) = 0 \text{ for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2 = 0 \tag{2.3.32}$$

From the integration formula

$$\int \frac{dt}{at^2+bt+c} = \frac{2}{\sqrt{4ac-b^2}}\arctan(\frac{2at+b}{\sqrt{4ac-b^2}}) + \text{constant}$$

for real constants a, b and c such that $4ac-b^2>0$,
we find that

$$F_2^{(k)}(\xi,\eta) = \frac{l^{(k)}[n_x^{(k)}(x^{(k)}-\xi)+n_y^{(k)}(y^{(k)}-\eta)]}{\pi\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}}$$
$$\times[\arctan(\frac{2A^{(k)}+B^{(k)}(\xi,\eta)}{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}})$$
$$-\arctan(\frac{B^{(k)}(\xi,\eta)}{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}})$$
$$\text{for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2 > 0. \tag{2.3.33}$$

If $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2 = 0$, we may write

$$A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta) = A^{(k)}(t+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}})^2.$$

Thus,

$$F_1^{(k)}(\xi,\eta)=\frac{l^{(k)}}{4\pi}\int_0^1 \ln[A^{(k)}(t+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}})^2]dt$$

$$\text{for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0.$$

(2.3.34)

Now if (ξ,η) lies on a smooth part of $D^{(k)}$, the integral in Eq. (2.3.34) is improper, as its integrand is not well defined at the point $t=t_0\equiv -B^{(k)}(\xi,\eta)/(2A^{(k)})\in(0,1)$. Strictly speaking, the integral should then be interpreted in the Cauchy principal sense, that is to evaluate it, we have to integrate over $[0,t_0-\varepsilon]\bigcup[t_0+\varepsilon,1]$ instead of $[0,1]$ and then let $\varepsilon\to 0$ to obtain its value. However, in this case, it turns out that the limits of integration $t=t_0-\varepsilon$ and $t=t_0+\varepsilon$ eventually do not contribute anything to the integral. Thus, for $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0$, the final analytical formula for $F_1^{(k)}(\xi,\eta)$ is the same irrespective of whether (ξ,η) lies on $D^{(k)}$ or not. If (ξ,η) lies on $D^{(k)}$, we may ignore the singular behaviour of the integrand and apply the fundamental theorem of integral calculus to evaluate the definite integral in Eq. (2.3.34) directly over [0,1], (Bnerjiee[16]).

The integration required in Eq. (2.3.34) can be easily done to give

$$F_1^{(k)}(\xi,\eta)=\frac{l^{(k)}}{2\pi}\left\{\ln(l^{(k)})+\left(1+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}\right)\ln\left|1+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}\right|\right.$$
$$\left.-\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}\ln\left|\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}\right|-1\right\}$$

$$\text{for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0.$$

(2.3.35)

Using

$$\int\ln(at^2+bt+c)dt=t[\ln(a)-2]+(t+\frac{b}{2a})\ln[t^2+\frac{b}{a}t+\frac{c}{a}]$$
$$+\frac{1}{a}\sqrt{4ac-bt}\,\arctan(\frac{2at+b}{\sqrt{4ac-b^2}})+\text{constant}$$

for real constants a,b and c such that $4ac-b^2>0$,

we obtain

$$F_1^{(k)}(\xi,\eta)=\frac{l^{(k)}}{4\pi}\{2[\ln(l^{(k)}-1)]-\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}\ln\left|\frac{E^{(k)}(\xi,\eta)}{A^{(k)}}\right|$$
$$+(1+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}})\ln\left|1+\frac{B^{(k)}(\xi,\eta)}{A^{(k)}}+\frac{E^{(k)}(\xi,\eta)}{A^{(k)}}\right|$$

$$+\frac{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}}{A^{(k)}}$$
$$\times[\arctan(\frac{2A^{(k)}+B^{(k)}(\xi,\eta)}{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}})$$
$$-[\arctan(\frac{B^{(k)}(\xi,\eta)}{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}})]]\}$$
$$\text{for }\ 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2>0.$$

(2.3.36)

2.4 Example

In order to illustrate the performance of the BEM proposed, we can consider the exact solution of this particular boundary value problem as follows

$$T=\frac{\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}$$

(2.4.37)

There is a significant improvement in the accuracy of the numerical results when the number of boundary elements used is increased from 60 to 120.
Now, we take the solution domain as the square region $0<x<1,\ 0<y<1$.

The boundary conditions are

$$\left.\begin{array}{ll} T=0 & \text{on } x=0 \\ T=\cos(\pi y) & \text{on } x=1 \end{array}\right\}\text{for }\ 0<y<1$$

$$\frac{\partial T}{\partial n}=0 \ \text{ on }\ y=0 \text{ and } y=1 \text{ for } 0<x<1$$

(2.4.38)

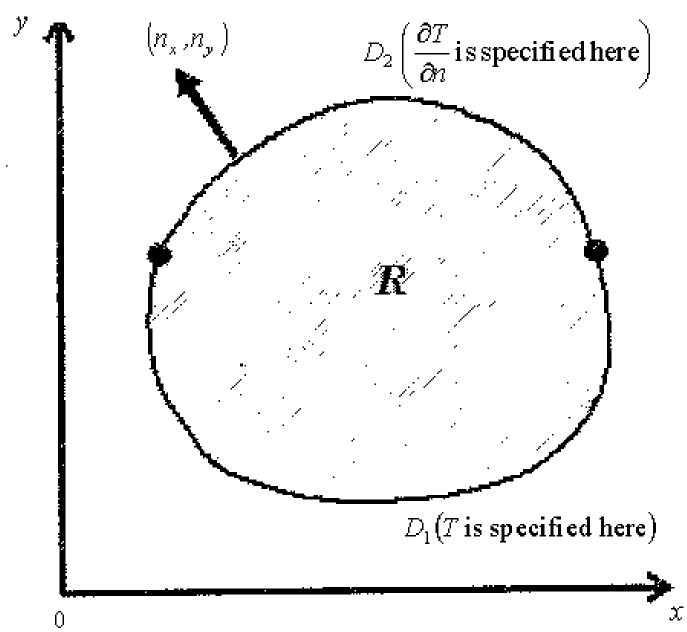
The sides of the square are discretized into boundary elements of equal length. To do this, we choose N evenly spaced out points on the sides as follows $(x^{(1)},y^{(1)})$, $(x^{(2)},y^{(2)})$,......, $(x^{(N-1)},y^{(N-1)}),(x^{(N)},y^{(N)})$ and $(x^{(N+1)},y^{(N+1)})$ arranged in counter clockwise order on the boundary of the solution domain.

Now we will compare the numerical values of T at various interior points obtained using 60 and 120 boundary elements with the exact solution in table (2.1).

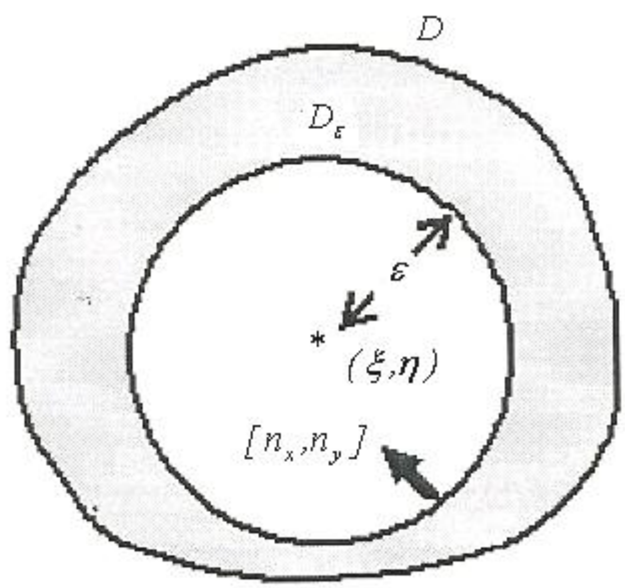
Points	BEM N=60	BEM N=120	FDM	Exact
(0.1,0.2)	0.0226	0.0224	0.0214	0.0224
(0.1,0.3)	0.0165	0.0163	0.0143	0.0163
(0.1,0.4)	0.0086	0.0085	0.0012	0.0085
(0.5,0.2)	0.1622	0.1614	0.1530	0.1612
(0.5,0.3)	0.1180	0.1173	0.1169	0.1171
(0.5,0.4)	0.0621	0.0617	0.0562	0.0616
(0.9,0.2)	0.5895	0.5899	0.5870	0.5899
(0.9,0.3)	0.4283	0.4286	0.4198	0.4286
(0.9,0.4)	0.2251	0.2253	0.2196	0.2253

Table (2.1)

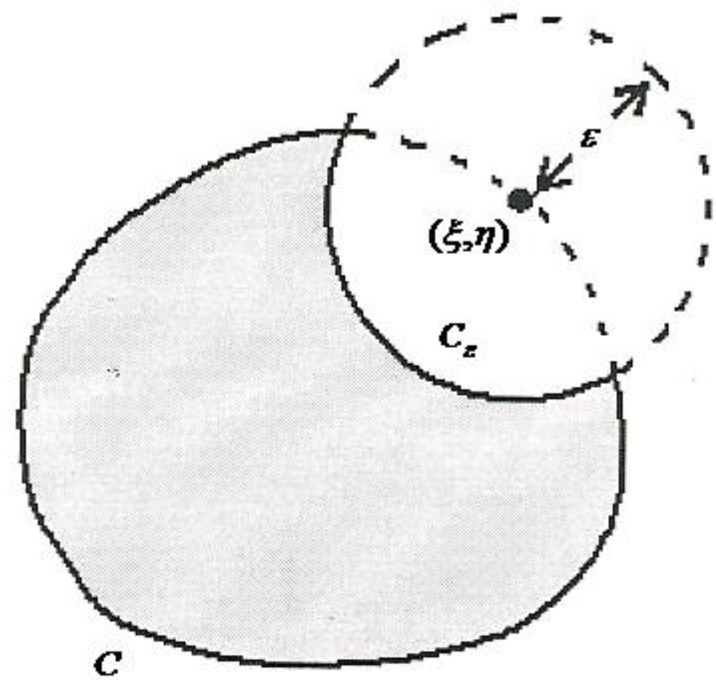
The result, we found that the boundary element solution agrees quite well with the exact solution. And more suitable than the result we obtained from the finite difference method.



(Fig 2.1)


D
D_ε
ε
*
(ξ,η)
$[n_x,n_y]$

(Fig 2.2


ε
(ξ,η)
C_ε
C

(Fig 2.3)

CHAPTER (III)

(Boundary Element Solution of Non Steady-State Temperature Distribution in Homogeneous Media)

3.1 Introduction

The application of boundary integral equation (BIE) formulation and boundary element methods (BEM) to inverse analysis have recently gained special attention in several fields of engineering. Of particular interest to this study is the recent monograph by Ingham and Yuan [37] dedicated to BIE for inverse analysis. Here, BIE applications to a broad class of inverse problems including identification of temperature dependent properties, the detection of surfaces cavities and flaws, inverse acoustic and electromagnetic scattering, crack-identification methods, and parameter identification in groundwater. Examples of BEM-based inverse formulation for identification of flaw and cavities can be found in Kassab et al. [39], Kassab and Pollard [40] and Lesnic et al. [42]. Chandra and Chan [24] explicitly determine design sensitivities using the BEM and direct differentiation in steady-state conduction-convection problems for modeling and optimization of thermal aspects of machining processes. Martin and Dulikravich [44] discussed a non-iterative algorithms to retrieve unknown heat sources and boundary conditioned in the ill-posed two dimensional Poisson problem using over-specified boundary conditions or internal temperature measurements. They use singular-value decomposition to regularize their formulation.

In this chapter

3.2 Formulation of the problem

The two-dimensional non steady-state temperature distribution in homogeneous media may be written in the following form

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \alpha \frac{\partial T}{\partial t}, \tag{3.2.1}$$

where α is a given positive constant, t denotes time and $T(x, y, t)$ is temperature.

For $t>0$, we are interested in solving the (heat) equation in Eq. (3.2.1) in the two-dimensional region R bounded by simple closed curve D (on the Oxy plane) subject to the initial- boundary conditions

$$T(x,y,0) = f(x,y) \text{ for } (x,y)\in R,$$
$$T(x,y,t) = g(x,y) \text{ for } (x,y)\in D_1 \text{ and } t>0,$$
$$\frac{\partial}{\partial n}[T(x,y,t)] = h(x,y) \text{ for } (x,y)\in D_2 \text{ and } t>0,$$

(3.2.2)

where f, g and h are suitably prescribed functions, D_1 and D_2 are non-intersecting curves such that $D_1 \cup D_2 = D$, $\partial T/\partial n = n_x \partial T/\partial x + n_y \partial T/\partial y$ and $[n_x, n_y]$ is the unit normal vector on D, pointing away from R.

The fundamental solution of the two-dimensional Laplace's equation as given (in Chapter II) by

$$\Psi(x,y;\xi,\eta) = \frac{1}{4\pi}\ln[(x-\xi)^2 + (y-\eta)^2],$$

(3.2.3)

may be used to convert Eq. (3.2.1) to an integro-differential equation given by

$$\lambda(\xi,\eta)T(\xi,\eta,t) - \iint_R \alpha\Psi(x,y;\xi,\eta)\frac{\partial}{\partial t}[T(x,y,t)dxdy$$
$$= \int_D [T(x,y,t)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y,t))]ds(x,y)$$
$$\text{for } (\xi,\eta)\in R\cup D,$$

(3.2.4)

where lies on a smooth part of

$$\lambda(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta)\notin R\cup D \\ 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } D \\ 1 & \text{if } (\xi,\eta)\in R \end{cases}$$

(3.2.5)

3.3 Boundary Element Procedure

Using the same technique as in sections 1.1 and 1.3 (see chapter I)

In this section, we show how Eq. (3.2.4) may be used to obtain a boundary element procedure for the numerical solution of the initial boundary value problem defined by Eqs. (3.2.1) and (3.2.2). The entire solution domain into many tiny cells, we apply the dual-reciprocity method to convert the domain integral in Eq. (3.2.4) approximately into a line integral over the boundary D. In implementing the boundary element procedure, only the boundary D has to be discretized into elements. The unknowns of the boundary element formulation here do not involve only the yet to be determined values of T or $\partial T/\partial n$ on the boundary elements but also those of T at selected collocation points in the interior of R.

The dual-reciprocity boundary element approach approximately reduces Eq. (3.2.4) into a system of linear equations containing unknown functions of time. First order time derivatives of some of the unknown functions are also present in the system. Several approaches may be employed for solving the system of linear algebraic-differential equations. The approach used here is to approximate the first order time derivatives using a finite difference formula, so that the initial-boundary value problem can be formulated as systems of linear algebraic equations to be solved at consecutive time levels separated by a small time-step. For the approach to work well, the boundary element procedure used to obtain the numerical solution must be sufficiently accurate. For this reason, we use the discontinuous linear elements for the approximation made on the boundary.

Eq. (3.2.4) may be written approximately as (Ang [13])

$$\lambda(\xi,\eta)T(\xi,\eta,t) - \alpha\sum_{j=1}^{M}\frac{\partial}{\partial t}[T(x,y,t)]\Bigg|_{(x,y)=(a^{(j)},b^{(j)})}\sum_{m=1}^{M}\omega^{(mj)}\Phi(\xi,\eta;a^{(m)},b^{(m)})$$
$$= \int_D [T(x,y,t)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)) - \Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y,t))]ds(x,y)$$
$$\text{for } (\xi,\eta)\in R\cup D, \tag{3.3.6}$$

where $(a^{(1)},b^{(1)}),(a^{(2)},b^{(2)}),\ldots(a^{(M-1)},b^{(M-1)})$ and $(a^{(M)},b^{(M)})$ are selected collocation points in $R\cup D$ and

$$\Phi(\xi,\eta;a,b) = \lambda(\xi,\eta)\chi(\xi,\eta;a,b)$$
$$+\int_D [\Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\chi(\xi,\eta;a,b))$$
$$-\chi(\xi,\eta;a,b)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)]ds(x,y),$$
$$\sum_{j=1}^{M}\omega^{(kj)}\rho(a^{(j)},b^{(j)};a^{(m)},b^{(m)}) = \begin{cases}1 & \text{if} \quad k=m,\\ 0 & \text{if} \quad k\neq m,\end{cases}$$
$$\rho(x,y;a,b) = 1 + r^2(x,y;a,b) + r^3(x,y;a,b),$$
$$\chi(x,y;a,b) = \frac{1}{4}r^2(x,y;a.b) + \frac{1}{16}r^4(x,y;a,b) + \frac{1}{25}r^5(x,y;a,b),$$
$$r(x.y;a.b) = \sqrt{(x-a)^2+(y-b)^2}. \tag{3.3.7}$$

We approximate the boundary integral in Eq. (3.3.6) using the discontinuous linear elements as detailed in W. T. Ang [13]. To do this, we discretize D into N straight line elements $D^{(1)}, D^{(2)}, \ldots, D^{(N-1)}$ and $D^{(N)}$. The endpoint of $k-th$ element $D^{(k)}$ are $(x^{(k)}, y^{(k)})$ and $(x^{(k+1)}, y^{(k+1)})$. (Note that $(x^{(N+1)}, y^{(N+1)}) = (x^{(1)}, y^{(1)})$.) Two points $(\xi^{(k)}, \eta^{(k)})$ and $(\xi^{(N+k)}, \eta^{(N+k)})$ at a distance of $\tau l^{(k)}$ from $(x^{(k)}, y^{(k)})$ and $(x^{(k+1)}, y^{(k+1)})$ respectively, where τ is a positive number such that $0 < \tau < 1/2$ and $l^{(k)}$ is the length of $D^{(k)}$, are chosen on $D^{(k)}$.

For discontinuous linear elements, we make the approximation

$$T(x,y,t) \cong \frac{[s(x,y)-(1-\tau)l^{(k)}]\hat{T}^{(k)}(t) - [s(x,y)-\tau\, l^{(k)}]\hat{T}^{(N+k)}(t)}{(2\tau-1)l^{(k)}} \quad \text{for } (x,y) \in D^{(k)}, \tag{3.3.8}$$

and

$$\frac{\partial}{\partial t}[T(x,y,t)] \cong \frac{[s(x,y)-(1-\tau)l^{(k)}]\hat{p}^{(k)}(t) - [s(x,y)-\tau\, l^{(k)}]\hat{p}^{(N+k)}(t)}{(2\tau-1)l^{(k)}} \quad \text{for } (x,y) \in D^{(k)}, \tag{3.3.9}$$

where $\hat{T}^{(k)}(t)$ and $\hat{T}^{(N+k)}(t)$ are the values of $T(x,y,t)$ at $(x,y) = (\xi^{(k)}, \eta^{(k)})$ and $(x,y) = (\xi^{(N+k)}, \eta^{(N+k)})$ respectively, $\hat{p}^{(k)}(t)$ and $\hat{p}^{(N+k)}(t)$ are the values of the normal derivative $\partial[T(x,y,t)]/\partial n$ at $(x,y) = (\xi^{(k)}, \eta^{(k)})$ and $(x,y) = (\xi^{(N+k)}, \eta^{(N+k)})$ respectively and

$$s(x,y) = \sqrt{(x-x^{(k)})^2 + (y-y^{(k)})^2} \quad \text{for } (x,y) \in D^{(k)}.$$

with Eqs. (3.3.8)and (3.3.9), we may write

$$\int_D [T(x,y,t)\frac{\partial}{\partial n}(\Psi(x,y;\xi,\eta)) - \Psi(x,y;\xi,\eta)\frac{\partial}{\partial n}(T(x,y,t))]ds(x,y)$$

$$\begin{aligned}
&\cong \sum_{k=1}^{N} \frac{1}{(2\tau-1)l^{(k)}} \{\hat{T}^{(k)}(t)[-(1-\tau)l^{(k)}F_2^{(k)}(\xi,\eta) + F_4^{(k)}(\xi,\eta)] \\
&+ \hat{T}^{(N+k)}(t)[\tau l^{(k)}F_2^{(k)}(\xi,\eta) - F_4^{(k)}(\xi,\eta)] \\
&- \hat{p}^{(k)}(t)[-(1-\tau)l^{(k)}F_1^{(k)}(\xi,\eta) + F_3^{(k)}(\xi,\eta)] \\
&- \hat{p}^{(N+k)}(t)[\tau l^{(k)}F_1^{(k)}(\xi,\eta) - F_3^{(k)}(\xi,\eta]\},
\end{aligned}$$

where analytical formulae for calculating $F_1^{(k)}(\xi,\eta), F_2^{(k)}(\xi,\eta), F_3^{(k)}(\xi,\eta)$ and $F_4^{(k)}(\xi,\eta)$ are given in Ang [13].

Note that $(a^{(1)},b^{(1)}),(a^{(2)},b^{(2)}),\ldots,(a^{(M-1)},b^{(M-1)})$ and $(a^{(M)},b^{(M)})$ in Eq. (3.3.6) are M selected points that are well spaced out in the region $R\bigcup D$. Taking $M=2N+L$, we choose the first $2N$ of these points to be those on the boundary elements given by $(\xi^{(k)},\eta^{(k)})$ and $(\xi^{(N+k)},\eta^{(N+k)})$ for $k=1,2,\ldots,N$. The remaining L points denoted by $(\xi^{(2N+1)},\eta^{(2N+1)})$, $(\xi^{(2N+2)},\eta^{(2N+2)})$,..., $(\xi^{(2N+l-1)},\eta^{(2N+l-1)})$ and $(\xi^{(2N+L)},\eta^{(2N+L)})$ are chosen points in the interior of R.

Below Eq. (3.3.9) we have already points defined $\hat{T}^{(n)}(t)=T(\xi^{(n)},\eta^{(n)},t)$ for $n=1,2,\ldots,2N$. view of the L selected points inside R, we now extend the definition to include $n=2N+1$, $2N+2$, ..., $2N+L$.
Eq. (3.3.6) may now be approximately written as

$$
\begin{aligned}
&\lambda(\xi,\eta)T(\xi,\eta,t)-\alpha\sum_{j=1}^{2N+L}\frac{d}{dt}[\hat{T}^{(j)}(t)]\sum_{m=1}^{2N+L}\omega^{(mj)}\Phi(\xi,\eta;\xi^{(m)},\eta^{(m)})\\
&=\sum_{k=1}^{N}\frac{1}{(2\tau-1)l^{(k)}}\{\hat{T}^{(k)}(t)[-(1-\tau)l^{(k)}F_2^{(k)}(\xi,\eta)+F_4^{(k)}(\xi,\eta)]\\
&+\hat{T}^{(N+k)}(t)[\tau\, l^{(k)}F_2^{(k)}(\xi,\eta)-F_4^{(k)}(\xi,\eta)]\\
&-\hat{p}^{(k)}(t)[-(1-\tau)l^{(k)}F_1^{(k)}(\xi,\eta)+F_3^{(k)}(\xi,\eta)]\\
&-\hat{p}^{(N+k)}(t)[\tau l^{(k)}F_1^{(k)}(\xi,\eta)-F_3^{(k)}(\xi,\eta]\},
\end{aligned}
\tag{3.3.10}
$$

We assume that either T or $\partial T/\partial n$ (not both) is specified across a given element. If T is specified on $D^{(k)}$ then $\hat{p}^{(k)}(t)$ and $\hat{p}^{(N+k)}(t)$ are unknown functions. Otherwise, if $\partial T/\partial n$ is specified on $D^{(k)}, \hat{T}^{(k)}(t)$ and $\hat{T}^{(N+k)}(t)$ are unknowns. At the selected interior points $(\xi^{(2N+1)},\eta^{(2N+1)})$, $(\xi^{(2N+2)},\eta^{(2N+2)})$, ..., $(\xi^{(2N+l-1)},\eta^{(2N+l-1)})$ and $(\xi^{(2N+L)},\eta^{(2N+L)})$, T is not known at all time except at $t=0$ [when it is given by the initial condition in Eq. (3.2.2)], that is $\hat{T}^{(2N+1)}(t)$, $\hat{T}^{(2N+2)}(t)$, ..., $\hat{T}^{(2N+l-1)}(t)$, $\hat{T}^{(2N+L)}(t)$ are unknown functions of t for $t>0$. Thus, there are $2N+L$ unknown functions of t on the right hand side of Eq. (3.3.10).

From Eq. (3.3.7), with the boundary D discretized into boundary elements, we may approximately evaluate $\Phi(\xi,\eta;a,b)$ using

$$
\begin{aligned}
&\Phi(\xi,\eta;a,b)=\lambda(\xi,\eta)\chi(\xi,\eta;a,b)\\
&-\sum_{k=1}^{N}\frac{1}{(2\tau-1)l^{(k)}}\Big\{\chi(\xi^{(k)},\eta^{(k)};a,b)[-(1-\tau)l^{(k)}F_2^{(k)}(\xi,\eta)+F_4^{(k)}(\xi,\eta)]\\
&+\chi(\xi^{(N+k)},\eta^{(N+k)};a,b)[\tau\, l^{(k)}F_2^{(k)}(\xi,\eta)-F_4^{(k)}(\xi,\eta)]\\
&-\frac{\partial}{\partial n}[\chi(\xi,\eta;a,b)]\Big|_{(\xi,\eta)=(\xi^{(k)},\eta^{(k)})}[-(1-\tau)l^{(k)}F_1^{(k)}(\xi,\eta)+F_3^{(k)}(\xi,\eta)]\\
&-\frac{\partial}{\partial n}[\chi(\xi,\eta;a,b)]\Big|_{(\xi,\eta)=(\xi^{(N+k)},\eta^{(N+k)})}[\tau\, l^{(k)}F_1^{(k)}(\xi,\eta)+F_3^{(k)}(\xi,\eta)]\Big\}.
\end{aligned}
$$

(3.3.11)

If we let (ξ,η) in Eq. (3.3.10) be given in turn by $(\xi^{(n)},\eta^{(n)})$ for $n=1,2,\ldots,2N+L$, we generate a system of $2N+L$ linear equations in $2N+L$ unknown functions of t that is we obtain

$$
\begin{aligned}
&\lambda(\xi^{(n)},\eta^{(n)})\hat{T}^{(n)}(t)-\alpha\sum_{j=1}^{2N+L}\mu^{(nj)}\frac{d}{dt}[\hat{T}^{(j)}(t)]\\
&=\sum_{k=1}^{N}\frac{1}{(2\tau-1)l^{(k)}}\Big\{\hat{T}^{(k)}(t)[-(1-\tau)l^{(k)}F_2^{(k)}(\xi^{(n)},\eta^{(n)})+F_4^{(k)}(\xi^{(n)},\eta^{(n)})]\\
&+\hat{T}^{(N+k)}(t)[\tau\, l^{(k)}F_2^{(k)}(\xi^{(n)},\eta^{(n)})-F_4^{(k)}(\xi^{(n)},\eta^{(n)})]\\
&-\hat{p}^{(k)}(t)[-(1-\tau)l^{(k)}F_1^{(k)}(\xi^{(n)},\eta^{(n)})+F_3^{(k)}(\xi^{(n)},\eta^{(n)})]\\
&-\hat{p}^{(N+k)}(t)[\tau l^{(k)}F_1^{(k)}(\xi^{(n)},\eta^{(n)})-F_3^{(k)}(\xi^{(n)},\eta^{(n)})]\Big\},\\
&\qquad\qquad\qquad\text{for}\quad n=1,2,\ldots,2N+L,
\end{aligned}
$$

(3.3.12)

where

$$
\mu^{(nj)}=\sum_{m=1}^{2N+L}\omega^{(mj)}\Phi(\xi^{(n)},\eta^{(n)};\xi^{(m)},\eta^{(m)}.
$$

(3.3.13)

Note that $\lambda(\xi^{(n)},\eta^{(n)})=1/2$ for $n=1,2,\ldots,2N$ and $\lambda(\xi^{(n)},\eta^{(n)})=1$ for $n=2N+1$ $,2N+2$ $,\ldots,2N+L$. The constants $\omega^{(mj)}$ are determined from Eq. (3.3.7) by letting $M=2N+L$ and $(a^{(n)},b^{(n)})=(\xi^{(n)},\eta^{(n)})$ for $n=1,2,\ldots,2N+L.$

If we are able to solve Eq. (3.3.12) together with the initial-boundary conditions in Eq. (3.2.2), then we have determined T numerically (for the initial-boundary value problem described in Section 3.2) at $2N+L$ selected points in $R\bigcup D.$

Now we will describe a time-stepping approach for solving Eq. (3.3.12) together with the initial-boundary conditions in Eq. (3.2.2).

The function $\hat{T}^{(j)}(t)$ and its first order derivative are approximated using

$$\hat{T}^{(j)}(t) \cong \frac{1}{2}[\hat{T}^{(j)}(t+\frac{1}{2}\Delta t)+\hat{T}^{(j)}(t-\frac{1}{2}\Delta t)]$$

$$\frac{d}{dt}[\hat{T}^{(j)}(t)] \cong \frac{1}{\Delta t}[\hat{T}^{(j)}(t+\frac{1}{2}\Delta t)+\hat{T}^{(j)}(t-\frac{1}{2}\Delta t)], \tag{3.3.14}$$

where Δt is a small (positive) time-step. The errors in the approximations in Eq. (3.3.14) have magnitudes which are of the order $O([\Delta t]^2)$.

Substitution of Eqs. (3.3.14) into Eq. (3.3.12) yields

$$\begin{aligned}
&\frac{1}{2}\lambda(\xi^{(n)},\eta^{(n)})[\hat{T}^{(n)}(t+\frac{1}{2}\Delta t)+\hat{T}^{(n)}(t-\frac{1}{2}\Delta t)]\\
&-\frac{\alpha}{\Delta t}\sum_{j=1}^{2N+L}\mu^{(nj)}[\hat{T}^{(j)}(t+\frac{1}{2}\Delta t)-\hat{T}^{(j)}(t-\frac{1}{2}\Delta t)]\\
&=\sum_{k=1}^{N}\frac{1}{(2\tau-1)l^{(k)}}\left\{\frac{1}{2}[\hat{T}^{(k)}(t+\frac{1}{2}\Delta t)+\hat{T}^{(k)}(t-\frac{1}{2}\Delta t)]\right.\\
&\times-(1-\tau)\,l^{(k)}F_2^{(k)}(\xi^{(n)},\eta^{(n)})-F_4^{(k)}(\xi^{(n)},\eta^{(n)})]\\
&+\frac{1}{2}[\hat{T}^{(N+k)}(t+\frac{1}{2}\Delta t)+\hat{T}^{(N+k)}(t-\frac{1}{2}\Delta t)]\\
&\times[\tau\,l^{(k)}F_2^{(k)}(\xi^{(n)},\eta^{(n)})+F_4^{(k)}(\xi^{(n)},\eta^{(n)})]\\
&-\hat{p}^{(k)}(t)[-(1-\tau)l^{(k)}F_1^{(k)}(\xi^{(n)},\eta^{(n)})+F_3^{(k)}(\xi^{(n)},\eta^{(n)})]\\
&\left.-\hat{p}^{(N+k)}(t)[\tau l^{(k)}F_1^{(k)}(\xi^{(n)},\eta^{(n)})-F_3^{(k)}(\xi^{(n)},\eta^{(n)}]\right\},\\
&\qquad\qquad\qquad\qquad\text{for}\quad n=1,2,\ldots,2N+L,
\end{aligned} \tag{3.3.15}$$

If we assume that $\hat{T}^{(j)}(t-\frac{1}{2}\Delta t)$ $(j=1,2,\ldots,2N+L)$ are known then Eq. (3.3.15) constitutes a system of $2N+L$ linear algebraic equations containing $2N+L$ unknowns. There are $2N$ unknowns on the boundary. They are given by $\hat{T}^{(k)}(t+\frac{1}{2}\Delta t)$ and $\hat{T}^{(N+k)}(t+\frac{1}{2}\Delta t)$ if $\partial T/\partial n$ is specified on the boundary element $D^{(k)}$, or by $\hat{p}^{(k)}(t)$ and $\hat{p}^{(N+k)}(t)$ if T is known on $D^{(k)}$. The remaining unknowns are the value of T at the L chosen interior points, as given by $\hat{T}^{(j)}(t+\frac{1}{2}\Delta t)$ for $j=2N+1,\ 2N+2\ ,\ \cdots\ ,\ 2N+L$.

Eq. (3.3.15) may be solved at consecutive time levels $t=\frac{1}{2}\Delta t,\frac{3}{2}\Delta t,\frac{5}{2}\Delta t,\ldots,$ as follows.

If we let $t=\frac{1}{2}\Delta t$, we find that $\hat{T}^{(j)}(t-\frac{1}{2}\Delta t)=\hat{T}^{(j)}(0)$ $(j=1,2,\ldots,2N+1)$. For $j=1,2,\ldots,L$ it is obvious that $\hat{T}^{(2N+j)}(0)$ can be determined directly from the initial condition in Eq. (3.2.2) that is $\hat{T}^{(2N+j)}(0)=f(\xi^{(2N+j)},\eta^{(2N+j)})$. If there is no discontinuity between the value of T in R at $t=0$ and that specified on D_1 for $t>0$, we may extend the initial condition to include all the points on the boundary and take $\hat{T}^{(m)}(0)=f(\xi^{(m)},\eta^{(m)})$ for $m=1,2,\ldots,2N$. If a discontinuity exists, and if T is known on $D^{(k)}$, then the known T on $D^{(k)}$ for $t>0$ is extended to include $t=0$, so that $\hat{T}^{(k)}(0)$ and $\hat{T}^{(N+k)}(0)$ are respectively given by $g(\xi^{(k)},\eta^{(k)},0)$ and $g(\xi^{(N+k)},\eta^{(N+k)},0)$ [instead of $f(\xi^{(k)},\eta^{(k)})$ and $f(\xi^{(N+k)},\eta^{(N+k)})$], in order to ensure that $d[\hat{T}^{(k)}(t)]/dt$ and $d[\hat{T}^{(N+k)}(t)]/dt$ are well defined at $t=0^{+}$. With $\hat{T}^{(n)}(0)$ known for $n=1,2,\ldots,2N+L$, we can solve Eq. (3.3.15) for the unknown given by either $\hat{T}^{(m)}(\Delta t)$ or $\hat{p}^{(m)}(\frac{1}{2}\Delta t)$ for $m=1,2,\ldots,2N$, and by $\hat{T}^{(2N+j)}(\Delta t)$ for $j=1,2,\ldots,L$. Once these unknowns are determined, we can go to the next time level $t=\frac{3}{2}\Delta t$ and solve Eq. (3.3.15) again for either $\hat{T}^{(m)}(2\Delta t)$ or $\hat{p}^{(m)}(\frac{3}{2}\Delta t)$ for $m=1,2,\ldots,2N$ and $\hat{T}^{(2N+j)}(2\Delta t)$ for $j=1,2,\ldots,L$. We may repeat the procedure at $t=\frac{5}{2}\Delta t$, $\frac{7}{2}\Delta t$, … to find the unknowns at higher time levels.

We may rewrite Eq. (3.3.15) as (Stehfest [48])

$$\sum_{k=1}^{N}(a^{(nk)}z^{k}(t)+a^{(n[N+k])}z^{(N+k)}(t))+\sum_{j=1}^{L}\ a^{(n[2N+j])}z^{(2N+j)}(t))$$
$$=\sum_{k=1}^{N}[b^{(nk)}(t)+c^{(nk)}(t)]$$
$$+\sum_{j=1}^{2N+L}[\frac{\alpha}{\Delta t}\mu^{(nj)}+\frac{1}{2}\delta^{(nj)}\lambda(\xi^{(j)},\eta^{(j)})]\hat{T}^{(j)}(t-\frac{1}{2}\Delta t) \qquad \text{for } n=1,2,\ldots,2N+L, \tag{3.3.16}$$

where

$$a^{(nk)} = \begin{cases} (2\tau-1)^{-1}[(1-\tau)F_1^{(k)}(\xi^{(n)},\eta^{(n)}) \\ -[l^{(k)}]^{-1}F_3^{(k)}(\xi^{(n)},\eta^{(n)})] & \text{if } T \text{ is given on } D^{(k)} \\ \\ (4\tau-2)^{-1}[-(1-\tau)F_2^{(k)}(\xi^{(n)},\eta^{(n)}) \\ +[l^{(k)}]^{-1}F_4^{(k)}(\xi^{(n)},\eta^{(n)})] & \text{if } \partial T/\partial n \text{ is given on } D^{(k)} \\ -\frac{1}{4}\delta^{(nk)}\alpha\mu^{(nk)}(\Delta t)^{-1} \end{cases}$$

$$a^{(n[N+k])} = \begin{cases} (2\tau-1)^{-1}[-\tau F_1^{(k)}(\xi^{(n)},\eta^{(n)}) \\ +[l^{(k)}]^{-1}F_3^{(k)}(\xi^{(n)},\eta^{(n)})] & \text{if } T \text{ is given on } D^{(k)} \\ \\ (4\tau-2)^{-1}[\tau F_2^{(k)}(\xi^{(n)},\eta^{(n)}) \\ -[l^{(k)}]^{-1}F_4^{(k)}(\xi^{(n)},\eta^{(n)})] & \text{if } \partial T/\partial n \text{ is given on } D^{(k)} \\ -\frac{1}{4}\delta^{([n-N]k)}\alpha\mu^{(n[N+k])}(\Delta t)^{-1} \end{cases}$$

$$a^{(n[2N+j])} = \frac{\alpha}{\Delta t}\mu^{(n[2N+j])} - \frac{1}{2}\delta^{[n-2N]j} \quad \text{for } j = 1,2,\ldots,L,$$

3.4 Example

In order to illustrate the performance of the BEM proposed, we can consider the exact solution of this particular boundary value problem as follows

$$T = 1 + e^{-\frac{\pi^2 t}{8}}\cos(\frac{\pi x}{4})\sin(\frac{\pi y}{4}) \tag{3.4.17}$$

There is a significant improvement in the accuracy of the numerical results when the number of boundary elements used is increased from 60 to 120.
Now, we take the solution domain as the square region $0 < x < 1,\ 0 < y < 1$.
The initial and boundary conditions are

$$T = 1 + \cos(\frac{\pi x}{4})\sin(\frac{\pi y}{4}) \quad \text{at } t = 0 \text{ for } 0 < x < 1,\ 0 < y < 1,$$

$$T = 1 + \exp(-\frac{\pi^2 t}{8})\sin(\frac{\pi y}{4}) \quad \text{on } x = 0 \text{ for } 0 < y < 1 \text{ and } t > 0,$$

$$T = 1 + \frac{1}{\sqrt{2}}\exp(-\frac{\pi^2 t}{8})\sin(\frac{\pi y}{4}) \quad \text{on } x = 1 \text{ for } 0 < y < 1 \text{ and } t > 0,$$

$$\frac{\partial T}{\partial n} = -\frac{\pi}{4}\exp(-\frac{\pi^2 t}{8})\cos(\frac{\pi y}{4}) \text{ on } y=0 \text{ for } 0<x<1 \text{ and } t>0,$$
$$\frac{\partial T}{\partial n} = \frac{\pi}{4\sqrt{2}}\exp(-\frac{\pi^2 t}{8})\cos(\frac{\pi x}{4}) \text{ on } y=1 \text{ for } 0<x<1 \text{ and } t>0,$$
(3.4.18)

Now We will compare the numerical values of T at various interior points at $t=1$ obtained using 60 and 120 boundary elements with the exact solution in table (3.1).

Points	BEM N=60, $t=1$	BEM N=120, $t=1$	FDM $t=1$	Exact $t=1$
(0.1,0.2)	1.0458	1.0454	1.0437	1.0454
(0.1,0.3)	1.0683	1.0677	1.0672	1.0678
(0.1,0.4)	1.0902	1.0898	1.0886	1.0897
(0.5,0.2)	1.0419	1.0422	1.0410	1.0421
(0.5,0.3)	1.0624	1.0629	1.0598	1.0628
(0.5,0.4)	1.0826	1.0832	1.0769	1.0831
(0.9,0.2)	1.0350	1.0346	1.0312	1.0346
(0.9,0.3)	1.0520	1.0517	1.0493	1.0517
(0.9,0.4)	1.0690	1.0684	1.0682	1.0684

Table (3.1)

The result, we found that the boundary element solution agrees quite well with the exact solution. And more suitable than the result we obtained from the finite difference method.

CHAPTER (IV)

(Boundary element solution of steady-state temperature distribution in non-homogeneous media)

4.1 Introduction

Many modern industrial materials, for instance functionally gradient materials, to meet ever-increasing demands placed on materials by modern technologies such as the single stage to orbit plan, ceramic engines, and advanced turbomachinary components. There are also many naturally occurring materials such as sedimentary rock and wood that exhibit material heterogeneity. Practical issues related to analysis of non-homogeneous media via the so-called ‘homogenization’ or effective statistical macroscopic description of thermal conductivity is reviewed in Furmanski [33]. Abd-Alla, et al. [8] studied magneto-thermoelastic problem in non-homogeneous isotropic cylinder. Al-Huniti [11], Barletta and Zanchini [17] discussed hyperbolic heat conduction equation, analysis in heterogeneous media thus finds much importance in engineering practice. However, analytical solutions of this problem are truly challenging due to the variable-coefficient partial differential equations arising in isotropic analysis and the presence of cross-derivatives of the dependent variables arising in the governing equation in anisotropic media. Various approaches were proposed for particular non-homogeneous isotropic media.

4.2 Formulation of the problem

In non-dimensional form the two-dimensional steady-state temperature distribution in non-homogeneous media may be written as follows

$$\frac{\partial}{\partial x_i}(\lambda_{ij}\frac{\partial T}{\partial x_j}) = 0 \text{ in } R, \tag{4.2.1}$$

Subject to

$$T(x,y) = g(x,y) \quad \text{for } (x,y) \in D_{1,}$$
$$q(x,y) = h(x,y) \quad \text{for } (x,y) \in D_{2,} \tag{4.2.2}$$

where x_1 and x_2 are the same dimensionless expressions x and y respectively. R is a two-dimensional region bounded by a simple closed curve D on the $0xy$ plane, $T(x,y)$ is the unknown function to be determined, λ_{ij} are non-negative coefficients satisfying the symmetry property $\lambda_{ij} = \lambda_{ij}$ are the strict inequality $\lambda_{12}^2 - \lambda_{11}\lambda_{22} < 0$ at all points in the region $R \cup D$, D_1 and D_2 are non-intersecting curves such that $D_1 \cup D_2 = D$, $q(x,y) = \lambda_{ij}(x,y)n_i(x,y)\partial T / \partial x_j$, $n_i(x,y)$ on D and g , h are suitably prescribed functions. If q is specified at all points on D then, to ensure compatibility with (4.2.1), the function q in (4.2.2) is required to satisfy

$$\oint_D h(x,y)ds(x,y) = 0 \tag{4.2.3}$$

In the present chapter, we consider the case in which coefficients of the non-homogenous anisotropic media take the form

$$\lambda_{ij}(x,y) = \lambda_{ij}^{(0)} u(x,y), \tag{4.2.4}$$

where u is a given positive function that can be partially differentiated at least twice with respect to x_i and $\lambda_{ij}^{(0)}$ are non-negative constants satisfying $\lambda_{ij}^{(0)} = \lambda_{ji}^{(0)}$ and $[\lambda_{12}^{(0)}]^2 - \lambda_{11}^{(0)}\lambda_{22}^{(0)} < 0$. After using a substitution to re-write (4.2.1) in a suitable form, we employ the fundamental solution, for the corresponding homogenous anisotropic media, which takes the form of a simple logarithmic function, to derive an integral formulation for the BVM under consideration. With such a fundamental solution, the integral formulation inevitably contains a domain integral over the region R. To use the formulation for deriving a BEM for the numerical solution of the BVP, we apply the dual-reciprocity method (DRM) proposed by Brebbia and Nardini [19] to convert the domain integral into a line integral approximately. The DRM requires us to collocate at points in $R \cup D$ but the discretization of the region R into tiny elements is not needed. Thus, in the proposed approach solving numerically (4.2.1) and (4.2.2) with (4.2.4), only the curve boundary D has to be discretized. In the literature, the term 'dual-reciprocity boundary element method' (DRBEM) is used to describe such a BEM approach. The DRBEM outlined in the present chapter is applicable for physically suitable u given by any general function that varies spatially in a sufficiently smooth manner. To assess the applicability of method, it is used to solve some specific problems.

4.3. Boundary element procedure

Using the same technique as in sections 1.1 and 1.3 (see chapter I)

With the substitution

$$T(x,y) = \frac{1}{\sqrt{u(x,y)}}\omega(x,y) \tag{4.3.5}$$

we fined that (4.2.1) with (4.2.4) can be re-written as

$$\lambda_{ij}^{(0)} \frac{\partial^2 \omega}{\partial x_i \partial x_j} = k(x,y)\omega \tag{4.3.6}$$

where k is given by

$$k(x,y) = \frac{1}{\sqrt{u(x,y)}} \lambda_{ij}^{(0)} \frac{\partial^2}{\partial x_i \partial x_j}[\sqrt{u(x,y)}] \tag{4.3.7}$$

If we pretend that the right hand side of (4.3.6) is known, i.e. if we regard (4.3.6) as a Poisson's equation, we can apply the analysis in Clements [26] to derive the integral equation

$$\begin{aligned}\gamma(\xi,\eta)\omega(\xi,\eta) = &\iint_R k(x,y)\omega(x,y)\Psi(x,y,\xi,\eta)dxdy \\ &+ \oint_D [\Gamma(x,y,\xi,\eta)\omega(x,y) \\ &- \Psi(x,y,\xi,\eta)\lambda_{ij}^{(0)} n_i(x,y)\frac{\partial}{\partial x_j}\{\omega(x,y)\}]ds(x,y)\end{aligned} \tag{4.3.8}$$

where

$$\gamma(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta) \notin R \cup D \\ 1 & \text{if } (\xi,\eta) \in \mathrm{R} \\ a & \text{if } 0 < a < 1 \text{ if } (\xi,\eta) \in \mathrm{D} \end{cases},$$

and

$$\Psi(x,y,\xi,\eta) = \frac{1}{2\pi\sqrt{\lambda_{11}^{(0)}\lambda_{22}^{(0)} - [\lambda_{12}^{(0)}]^2}} \mathrm{Re}\{\ln(x - \xi + \tau[y - \eta])\}$$

$$\Gamma(x,y,\xi,\eta) = \frac{1}{2\pi\sqrt{\lambda_{11}^{(0)}\lambda_{22}^{(0)} - [\lambda_{12}^{(0)}]^2}} \mathrm{Re}\left\{\frac{L(x,y)}{(x - \xi + \tau[\eta - y])}\right\}$$

$$L(x,y) = (\lambda_{11}^{(0)} + \tau\lambda_{12}^{(0)})n_1(x,y) + (\lambda_{21}^{(0)} + \tau\lambda_{22}^{(0)})n_2(x,y)$$

$$\tau = \frac{-\lambda_{12}^{(0)} + i\sqrt{\lambda_{11}^{(0)}\lambda_{22}^{(0)} - [\lambda_{12}^{(0)}]^2}}{\lambda_{22}^{(0)}} \quad (i = \sqrt{-1}).$$

(4.3.9)

with (4.3.6), we can re-write the integral equation (4.3.10) as follows (Ang, et al. [15])

$$\gamma(\xi,\eta)\sqrt{u(\xi,\eta)}T(\xi,\eta)$$

$$= \iint_R k(x,y)\sqrt{u(x,y)}T(x,y)$$
$$\times \Psi(x,y,\xi,\eta)dxdy$$
$$+ \oint_D [\Gamma(x,y,\xi,\eta)\sqrt{u(x,y)}T(x,y)$$
$$-[T(x,y)\lambda_{ij}^{(0)} n_i(x,y)\frac{\partial}{\partial x_j}\{\sqrt{u(x,y)}\}$$
$$+\frac{q(x,y)}{\sqrt{u(x,y)}}]\Psi(x,y,\xi,\eta)]ds(x,y)$$

(4.3.10)

Notice that $q(x,y) = \lambda_{ij}(x,y)n_i(x,y)\partial T / \partial x_j$ (as defined earlier on).

In the following section, the integral equation (4.3.6) is used to derive a DRBEM for the numerical solution of the boundary value problem defined by (4.2.1) and (4.2.2) with λ_{ij} as given by (4.2.4).

For the DRBEM, let us discretize the curve D into N straight line (boundary) elements denoted $D^{(1)}, D^{(2)}, \cdots, D^{(N-1)}$ and $D^{(N)}$, i.e. we make the approximation:

$$D \cong D^{(1)} \cup D^{(2)} \cup \ldots \cup D^{(N-1)} \cup D^{(N)}.$$

(4.3.11)

As we shall see later on, the DRBEM requires us to collocate equations at points on the boundary D and in the interior of R. For this purpose, we select N points on the boundary D given by $(\xi^{(1)},\eta^{(1)})$, $(\xi^{(2)},\eta^{(2)})$, $\cdots$, $(\xi^{(N-1)},\eta^{(N-1)})$, and $(\xi^{(N)},\eta^{(N)})$, and L well-spaced out points in the interior of the region R as denoted by $(\xi^{(N+1)},\eta^{(N+1)})$, $(\xi^{(N+2)},\eta^{(N+2)})$, $\cdots$, $(\xi^{(N+L-1)},\eta^{(N+L-1)})$, and $(\xi^{(N+L)},\eta^{(N+L)})$. For convenience, for $p = 1,2,\cdots,N$, we take $(\xi^{(p)},\eta^{(p)})$, to be midpoint of the line element $D^{(p)}$.

To apply the dual-reciprocity method (DRM) of Brebbia and Nardini [19] to transform the domain integral in (4.3.6) into a line integral, we first make the approximation

$$k(x,y)\sqrt{u(x,y)}T(x,y) \cong \sum_{p=1}^{N+L} a^{(p)}\sigma^{(p)}(x,y)$$

(4.3.12)

where $a^{(p)}$ are constants to be determined and

$$\sigma^{(p)}(x,y) = 1 + \left([x-\xi^{(p)} + \mathrm{Re}\{\tau\}\{y-\eta^{(p)}\}]^2 + [\mathrm{Im}\{\tau\}\{y-\eta^{(p)}\}]^2\right) + \left([x-\xi^{(p)} + \mathrm{Re}\{\tau\}\{y-\eta^{(p)}\}]^2 + [\mathrm{Im}\{\tau\}\{y-\eta^{(p)}\}]^2\right)^{3/2}.$$

(4.3.13)

It should be noted that for $\lambda_{ij}^{(0)} = \delta_{ij}$ (Kronecker-delta) we find that $\tau = i$ and (4.3.13) reduces to give the local interpolating function suggested by Zhang and Zhu [51].

We can let (x,y) in (4.3.12) be given by $(\xi^{(m)},\eta^{(m)})$ for $m=1,2,\cdots,N+L,$ to set up a system of linear algebraic equations in $a^{(p)}$ which can be inverted to obtain

$$a^{(p)} = \sum_{m=1}^{N+L} \sqrt{u(\xi^{(m)},\eta^{(m)})}T^{(m)}k(\xi^{(m)},\eta^{(m)})\chi^{(mp)}$$

(4.3.14)

where $T^{(m)} = T(\xi^{(m)},\eta^{(m)})$ $(m=1,2,\cdots,N+L)$ and $\chi^{(mp)}$ are constants defined by

$$\sum_{m=1}^{N+L} \sigma^{(p)}(\xi^{(m)},\eta^{(m)})\chi^{(mr)} \begin{cases} 1 & \text{if} \quad p=r \\ 0 & \text{if} \quad p \neq r \end{cases}$$

(4.3.15)

Using (4.3.12) and (4.3.14) and applying the DRM, we find that the double integral in (4.3.10) can be approximately re-written as follows (Ang and Cooke [14])

$$\iint_R k(x,y)\sqrt{u(x,y)}T(x,y)\Psi(x,y,\xi,\eta)ds(x,y)$$
$$\cong \sum_{m=1}^{N+L} \sqrt{u(\xi^{(m)},\eta^{(m)})}T^{(m)}k(\xi^{(m)},\eta^{(m)}) \sum_{p=1}^{N+L} \chi^{(mp)}\Phi^{(p)}(\xi,\eta)$$

(4.3.16)

where

$$\Phi^{(p)}(\xi,\eta) = \gamma(\xi,\eta)\theta^{(p)}(\xi,\eta) + \oint_D \Psi(x,y,\xi,\eta)\beta^{(p)}(x,y)ds(x,y) - \oint_D \theta^{(p)}(x,y)\Gamma(x,y,\xi,\eta)ds(x,y)$$

(4.3.17)

With

$$\left(\frac{\lambda_{11}^{(0)}\lambda_{22}^{(0)}-[\lambda_{12}^{(0)}]^2}{\lambda_{22}^{(0)}}\right)\theta^{(p)}(x,y)$$
$$=\frac{1}{4}\left([x-\xi^{(p)}+\text{Re}\{\tau\}\{y-\eta^{(p)}\}]^2+[\text{Im}\{\tau\}\{y-\eta^{(p)}\}]^2\right)$$
$$+\frac{1}{16}\left([x-\xi^{(p)}+\text{Re}\{\tau\}\{y-\eta^{(p)}\}]^2+[\text{Im}\{\tau\}\{y-\eta^{(p)}\}]^2\right)^2$$
$$+\frac{1}{25}\left([x-\xi^{(p)}+\text{Re}\{\tau\}\{y-\eta^{(p)}\}]^2+[\text{Im}\{\tau\}\{y-\eta^{(p)}\}]^2\right)^{5/2}$$

(4.3.18)

and

$$\beta^{(p)}(x,y)=\lambda_{ik}^{(0)}n_i(x,y)\frac{\partial\theta^{(p)}}{\partial x_k}$$

(4.3.19)

The integral equation (4.3.10) together with (4.3.11) and (4.3.16) may used to derive

$$\gamma(\xi^{(n)},\eta^{(n)})\sqrt{u(\xi^{(n)},\eta^{(n)})}T^{(n)}$$
$$=\sum_{m=1}^{N+L}\sqrt{u(\xi^{(m)},\eta^{(m)})}T^{(m)}k(\xi^{(m)},\eta^{(m)})\sum_{p=1}^{N+L}\chi^{(pm)}\Phi^{(p)}(\xi^{(n)},\eta^{(n)})$$
$$+\sum_{m=1}^{N}\sqrt{u(\xi^{(m)},\eta^{(m)})}u^{(m)}\int_{D^{(m)}}\Gamma(x,y,\xi^{(n)},\eta^{(n)})ds(x,y)$$
$$-\sum_{m=1}^{N}[T^{(m)}\lambda_{ij}^{(0)}n_i^{(m)}\frac{\partial}{\partial x_j}\{\sqrt{u(x,y)}\}\Big|_{(x,y)=(\xi^{(m)},\eta^{(m)})}$$
$$+\frac{q^{(m)}}{\sqrt{u(\xi^{(m)},\eta^{(m)})}}]\int_{D^{(m)}}\Psi(x,y,\xi^{(n)},\eta^{(n)})ds(x,y)$$
$$\text{for } n=1,2,\cdots,N+L$$

(4.3.20)

where $q^{(m)}=q(\xi^{(m)},\eta^{(m)})$ $(m=1,2,\cdots,N)$ and $[n_1^{(m)},n_2^{(m)}]$ is the outward unit normal vector to $D^{(m)}$. Notice that, in deriving (4.3.20) we let (x,y) in (4.3.10) be given by $(\xi^{(m)},\eta^{(m)})$ for $n=1,2,\cdots,N+L,$ and in the integrands of the line integrals over $D^{(m)}$, we approximate the functions multiplied to $\Psi(x,y;\xi^{(n)},\eta^{(n)})$ and $\Gamma(x,y;\xi^{(n)},\eta^{(n)})$ as constants given by the values (of the functions) at the midpoint of $D^{(m)}$.

In view of the boundary conditions (4.2.2), either $T^{(m)}$ or $q^{(m)}$ (not both) is known for $m=1,2,\ldots,N$. Being the values of u at the interior collocation points $(\xi^{(N+1)},\eta^{(N+1)})$, $(\xi^{(N+2)},\eta^{(N+2)}),\cdots,$ $(\xi^{(N+L-1)},\eta^{(N+L-1)})$, and $(\xi^{(N+L)},\eta^{(N+L)})$, the constants $T^{(N+1)},T^{(N+2)},\cdots,T^{(N+L-1)}$ and $T^{(N+L)}$ are not known. Thus, the system (4.3.10) consists of $N+L$ linear algebraic equations which can be solved for $N+L$ unknown given by either $T^{(m)}$ or $q^{(m)}$ for $m=1,2,\cdots,N$ and T for $n=1,2,\cdots,L.$

4.4. Example

A 2-D steady-state heat conduction problem in an orthotropic heterogeneous medium is considered with a thermal conductivity taken as,

$$k(x,y)=\begin{bmatrix}2x+y+5 & 0\\ 0 & 3x+y+7\end{bmatrix}, \tag{4.4.21}$$

With this k, an exact temperature satisfying the governing equations is,

$$T(x,y)=4x^2+10xy-7y^2+20x+18y. \tag{4.4.22}$$

Points	BEM N=60	BEM N=120	FDM	Exact
(0.1,0.2)	5.5359	5.5572	5.5298	5.5600
(0.1,0.3)	7.0975	7.1087	7.0964	7.1100
(0.1,0.4)	8.4898	8.5189	8.4889	8.5200
(0.5,0.2)	15.2999	15.3210	15.2983	15.3200
(0.5,0.3)	17.2458	17.2689	17.2449	17.2700
(0.5,0.4)	19.0797	19.0800	19.0782	19.0800
(0.9,0.2)	26.3059	26.3598	26.2994	26.3600
(0.9,0.3)	28.6554	28.7095	28.6493	28.7100
(0.9,0.4)	30.9089	30.9157	30.9037	30.9200

Table (4.1)

The result, we found that the boundary element solution agrees quite well with the exact solution. And more suitable than the result we obtained from the finite difference method.

CHAPTER (V)

(Boundary element solution of non steady-state temperature distribution in non-homogeneous media)

5.1 Introduction

The increasing use of anisotropic material in engineering application has resulted in considerable research activity in this area in recent years. An understanding of thermally-induced stresses in anisotropic bodies is essential for a comprehensive study of their response due to an exposure to a temperature field, which may in turn occur in service or during the manufacturing stages. Our study in the present chapter is of fundamental importance to several disceiplines such as geophysics, earthquake engineering, geomechanics, composites, nondestructive testing, etc.

Recent technological advances allowed further a miniaturisation of electronic devices and an increase of their operating frequency. Unfortunately, both these factors augmented significantly the dissipated power density. Therefore, cooling problems occur now even in apparently low power applications. For this reason, still more and more products undergo in their design process a thermal simulation. Most commercial thermal simulators employ numerical methods for the solution of the heat equation. In order to obtain accurate results, these methods require a dense structure meshing, especially where the temperature gradient values are significant, which is time consuming. Thus, regarding the cost and time of a design process, analytical solutions providing explicit formulas, relating power dissipation to temperature rise, would be much more desirable, but usually they are difficult to find. In recent years, the BEM has also been found to be especially accurate and efficient in the analysis of thin elastic structures or materials. The assumption that the inertia terms may be omitted from the equations of motion holds good only when the variation in stresses, displacements and temperature with time are negligible, as we assumed for the displacement equation in the present chapter, but the heat equation is time dependent.

Abd-Alla, et al. [5] studied thermal stresses in a rotating orthotropic composite tubes. Abd-Alla, et al. [6, 7] discussed thermoelastic waves under the effect of initial stress. El-Naggar, et al. [29] proposed explicit finite difference scheme to obtain thermal stresses in an infinite slab. Also, they discussed transient thermal stresses in a rotating non-homogeneous orthotropic hollow cylinder. Linear and nonlinear boundary value problems were discussed by Barletta and Zanchini [17], and Nayfeh and Nasser [46] considered various mechanical problems coupled to electromagnetic effects through a large magnetic field.

The boundary element method (BEM), based on the boundary integral equation (BIE) formulation, is well known for its accuracy and efficiency in stress analysis.

The dual-reciprocity boundary element method was originally introduced by Brebbia and Nardini [19] for the numerical solution of dynamic problems in solid mechanics. The method has now been successfully extended to a wide range of heat diffusion problems in engineering. refer to Zhu, et al. [52].

This chapter presents a general treatment of the transient temperature distribution in a non-homogeneous anisotropic media. The heat conduction equation is solved by means of a boundary element method (BEM) and the numerical calculations are carried out for the temperature. The numerical and exact values show good agreement.

5.2 Formulation of the problem

In non-dimensional form the two-dimensional non steady state temperature distribution in non-homogeneous anisotropic media may be written in the following form (Fahmy [31])

$$\frac{\partial}{\partial x_i}(\lambda_{ij}\frac{\partial T}{\partial x_j}) = \rho c \frac{\partial T}{\partial t} \tag{5.2.1}$$

Subject to

$$\begin{aligned} &T(x,y,0) = f(x,y) \quad \text{for}\,(x,y)\in R\cup D,\\ &T(x,y,t) = g(x,y,t) \quad \text{for}\,(x,y)\in D_1, t>0,\\ &q(x,y,t) = h(x,y,t) \quad \text{for}\,(x,y)\in D_2, t>0. \end{aligned} \tag{5.2.2}$$

where x_1 and x_2 are the same dimensionless expressions x and y respectively, λ_{ij} are heat conductivity coefficients such that the symmetry relation $\lambda_{ij} = \lambda_{ji}$ is satisfied and the strict inequality $(\lambda_{12})^2 - \lambda_{11}\lambda_{22} < 0$ holds at all points in the solid, ρ is the density, c is the specific heat capacity of the solid and t is the dimensionless time. Also, f, g and h are suitably prescribed functions of x and y, D_1 and D_2 are non-intersecting curves such that $D = D_1 \cup D_2$.

In the present chapter, we consider the task of solving (5.2.1) and (5.2.2) for the case in which thermal conductivity coefficients are

$$\lambda_{ij}(x,y) = \lambda_{ij} u(x,y) \tag{5.2.3}$$

where u is a function that may be partially differentiated at least twice with respect to x and λ_{ij} are non-negative constants (the values of λ_{ij} in homogeneous matter).

5.3 Boundary element procedure

Using the same technique as in sections 1.1 and 1.3 (see chapter I)

We shall now outline a boundary element procedure for solving (5.2.1) subject to (5.2.2) in a two-dimensional region R bounded by a simple closed curve D.

From (5.2.1), (5.2.2) and (5.2.3) we obtain the integro-differential equation (Ang [12])

$$\begin{aligned}
&\gamma(\xi,\eta)\sqrt{u(\xi,\eta)}T(\xi,\eta,t) \\
&= \iint_R \left\{ k(x,y)T(x,y,t) + \frac{\rho c}{\sqrt{u(x,y)}} \frac{\partial}{\partial t}[T(x,y,t)] \right\} \Psi(x,y,\xi,\eta)dxdy \\
&+ \int_D \Gamma(x,y,\xi,\eta)\sqrt{u(x,y)}T(x,y,t) \\
&\quad - \left[\lambda_{ij} n_i(x,y)T(x,y,t) \frac{\partial}{\partial x_j} \sqrt{u(x,y)} \right. \\
&\left. - \frac{q(x,y,t)}{\sqrt{u(x,y)}} \right] \Psi(x,y,\xi,\eta)ds(x,y)
\end{aligned} \tag{5.3.4}$$

where

$$\gamma(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta) \notin R \cup D \\ 1 & \text{if } (\xi,\eta) \in R \\ a, \quad 0 < a < 1 & \text{if } (\xi,\eta) \in D \end{cases} \tag{5.3.5}$$

$$\begin{aligned}
&\Psi(x,y,\xi,\eta) \\
&= \frac{1}{2\pi\sqrt{\lambda_{11}\lambda_{22} - (\lambda_{12})^2}} \operatorname{Re}\{\ln[(x-\xi) + \tau(y-\eta)]\},
\end{aligned} \tag{5.3.6}$$

$$\Gamma(x,y,\xi,\eta)=\frac{1}{2\pi\sqrt{\lambda_{11}\lambda_{22}-(\lambda_{12})^2}}\operatorname{Re}\left[\hat{K}\right],$$
$$\hat{K}=\frac{(\lambda_{11}+\tau\lambda_{12})n_1(x,y)+(\lambda_{21}+\tau\lambda_{22})n_2(x,y)}{(x-\xi)+\tau(y-\eta)}$$

(5.3.7)

$$\tau=\frac{-\lambda_{12}+i\sqrt{\lambda_{11}\lambda_{22}-(\lambda_{12})^2}}{\lambda_{22}},\ (i=\sqrt{-1}),\ k=\lambda_{ij}\frac{\partial^2}{\partial x_i\partial x_j}(\sqrt{u(x,y)}),$$

(5.3.8)

The boundary D is discretized into N straight line elements denoted by $D^{(1)}, \mathrm{D}^{(2)},\ldots, D^{(N-1)}$ and $D^{(N)}$, and we employ discontinuous linear boundary elements (Clements [26]) to obtain the terms $\sqrt{u(x,y)}T(x,y,t)$, $T(x,y,t)\frac{\partial}{\partial x_j}\sqrt{u(x,y)}$ and $\frac{1}{\sqrt{u(x,y)}}q(x,y,t)$ in (5.3.4), where $(\eta_1^{(m)},\eta_2^{(m)})$ and $(\eta_1^{(N+m)},\eta_2^{(N+m)})$ on $D^{(m)}$ are chosen as:

$$\left.\begin{aligned}\eta_i^{(m)}&=a_i^{(m)}+r_0(b_i^{(m)}-a_i^{(m)})\\ \eta_i^{(N+m)}&=b_i^{(m)}+r_0(b_i^{(m)}-a_i^{(m)})\end{aligned}\right\}\quad \text{for a given}\quad r_0\in(0,\frac{1}{2}).$$

(5.3.9)

Hence, we have for $(x,y)\in D^{(m)}$

$$\sqrt{u(x,y)}T(x,y,t)\approx\sqrt{u(\eta_1^{(m)},\eta_2^{(m)})}T^{(m)}(t)\left[1-d^{(m)}(x,y)\right]$$
$$+\sqrt{u(\eta_1^{(N+m)},\eta_2^{(N+m)})}T^{(N+m)}(t)d^{(m)}(x,y),$$
$$\text{for } (x,y)\in D^{(m)}$$

(5.3.10)

$$T(x,y,t)\frac{\partial}{\partial x_j}\sqrt{u(x,y)}\approx\frac{\partial}{\partial x_j}\sqrt{u(x,y)}\Big|_{(x,y)=(\eta_1^{(m)},\eta_2^{(m)})}T^{(m)}(t)[1-d^{(m)}(x,y)]$$
$$+\frac{\partial}{\partial x_j}\sqrt{u(x,y)}\Big|_{(x,y)=(\eta_1^{(N+m)},\eta_2^{(N+m)})}T^{(N+m)}(t)d^{(m)}(x,y),$$
$$\text{for } (x,y)\in D^{(m)}$$

(5.3.11)

and

$$\frac{1}{\sqrt{u(x,y)}}q(x,y,t)\approx\frac{1}{\sqrt{u(\eta_1^{(m)},\eta_2^{(m)})}}q^{(m)}(t)\left[1-d^{(m)}(x,y)\right]$$

$$+\frac{1}{\sqrt{u(\eta_1^{(N+m)},\eta_2^{(N+m)})}}q^{(N+m)}(t)d^{(m)}(x,y).$$

(5.3.12)

where

$$d^{(m)}(x,y)=\frac{\sqrt{(x-a_1^{(m)})^2+(y-a_2^{(m)})^2}-r_0 l^{(m)}}{(1-2r_0)l^{(m)}}$$

(5.3.13)

We implement the dual-reciprocity boundary element method in (5.3.4), with considering the following approximation

$$k(x,y)T(x,y,t)+\frac{\rho c}{\sqrt{u(x,y)}}\frac{\partial}{\partial t}\left[T(x,y,t)\right]\approx\sum_{p=1}^{2N+L}a^{(p)}\sigma^{(p)}(x,y)$$

(5.3.14)

To obtain a system of $2N+L$ first order linear ordinary differential equations, where $a^{(p)}$ are constants to be determined and

$$\sigma^{(p)}(x,y)=1+\left(\left[x-\eta_1^{(p)}+\mathrm{Re}(\tau)(y-\eta_2^{(p)})\right]^2+\left[\mathrm{Im}(\tau)(y-\eta_2^{(p)})\right]^2\right)$$
$$+\left(\left[x-\eta_1^{(p)}+\mathrm{Re}(\tau)(y-\eta_2^{(p)}\right]^2+\left[\mathrm{Im}(\tau)(y-\eta_2^{(p)})\right]^2\right)^{3/2}$$

(5.3.15)

where $(\eta_1^{(1)},\eta_2^{(1)}),(\eta_1^{(2)},\eta_2^{(2)}),\ldots,(\eta_1^{(2N)},\eta_2^{(2N)})$ are the $2N$ points on the boundary elements as defined by (5.3.6) and $(\eta_1^{(2N+1)},\eta_2^{(2N+1)})$, $(\eta_1^{(2N+2)},\eta_2^{(2N+2)}),\ldots,(\eta_1^{(2N+L)},\eta_2^{(2N+L)})$ are L selected points in the interior of R.

We can let (x,y) in (5.3.14) be given by $(\eta_1^{(m)},\eta_2^{(m)})$ for $m=1,2,\ldots,$ $2N+L,$ to set up a system of linear algebraic equations in $a^{(p)}$. The algebraic equations can then be inverted to obtain

$$a^{(p)}=\sum_{m=1}^{2N+L}\left\{k(\eta_1^{(m)},\eta_2^{(m)})T^{(m)}(t)\right.$$
$$\left.+\frac{\rho c}{\sqrt{u(\eta_1^{(m)},\eta_2^{(m)})}}\frac{d}{dt}\left[T^{(m)}(t)\right]\right\}\chi^{(mp)}$$

(5.3.16)

where $T^{(m)}=T(\xi^{(1)},\eta^{(1)},t)\ (m=1,2,\ldots,2N+L)$ and $\chi^{(mp)}$ are constants defined by

$$\sum_{m=1}^{2N+L} \sigma^{(p)}(\eta_2^{(m)}, \eta_2^{(m)})\chi^{(mp)} = \begin{cases} 1 \text{ if } & p = r \\ 0 \text{ if } & p \neq r \end{cases}$$

(5.3.17)

Using (5.3.14) and (5.3.16) and applying the dual-reciprocity boundary element method, we find that the double integral in (5.3.4) can be approximately re-written as

$$\iint_R \left\{ k(x,y)T(x,y,t) + \frac{\rho c}{\sqrt{u(x,y)}} \frac{\partial}{\partial t}\left[T(x,y,t)\right] \right\} \Psi(x,y,\xi,\eta)dxdy$$

$$\approx \sum_{m=1}^{N+L} \left\{ k(\eta_2^{(m)}, \eta_2^{(m)})T^{(m)}(t) \right.$$

$$\left. + \frac{\rho c}{\sqrt{u(\eta_1^{(m)}, \eta_2^{(m)})}} \frac{d}{dt}\left[T^{(m)}(t)\right] \right\} \sum_{p=1}^{N+L} \chi^{(mp)} \Phi^{(p)}(\xi,\eta)$$

(5.3.18)

where

$$\Phi^{(p)}(\xi,\eta) = \gamma(\xi,\eta)\theta^{(p)}(\xi,\eta)$$

$$+ \int_D [\Psi(x,y,\xi,\eta)\beta^{(p)}(x,y) - \Gamma(x,y,\xi,\eta)\theta^{(p)}(x,y)]ds(x,y)$$

(5.3.19)

with

$$\left(\frac{\lambda_{11}\lambda_{22} - (\lambda_{12})^2}{\lambda_{22}} \right) \theta^{(p)}(x,y)$$

$$= \frac{1}{4}\left(\left[x - \eta_1^{(p)} + \text{Re}(\tau)(y - \eta_2^{(p)})\right]^2 + \left[\text{Im}(\tau)(y - \eta_2^{(p)})\right]^2 \right)$$

$$+ \frac{1}{16}\left(\left[x - \eta_1^{(p)} + \text{Re}(\tau)(y - \eta_2^{(p)})\right]^2 + \left[\text{Im}(\tau)(y - \eta_2^{(p)})\right]^2 \right)^2$$

$$+ \frac{1}{25}\left(\left[x - \eta_1^{(p)} + \text{Re}(\tau)(y - \eta_2^{(p)})\right]^2 + \left[\text{Im}(\tau)(y - \eta_2^{(p)})\right]^2 \right)^{5/2}$$

(5.3.20)

and

$$\beta^{(p)}(x,y) = \lambda_{ij} n_i(x,y) \frac{\partial \theta^{(p)}}{\partial x_j}.$$

(5.3.21)

The dual-reciprocity boundary element method approximately reduces the integro-differential equation into a system of $2N + L$ first order linear ordinary

differential equations to be solved subject to known values of $T^{(n)}(t)$ at $t=0$ as given by the initial temperature in (5.2.2).
To solve the system of ordinary differential equations, we approximate the nodal temperature $T^{(n)}(t)$ $(n=1,2,\ldots,2N+L)$ as follows (Stehfest [48])

$$T^{(n)}(t) \approx \sum_{l=1}^{M+1} T^{(n)}(t^{(l)}) \frac{\prod_{j=1,\, j\neq l}^{M+1} (t-t^{(j)})}{\prod_{\lambda=1, k\neq l}^{M+1} (t^{(l)}-t^{(\lambda)})}$$

(5.3.22)

where $t^{(*)} = t_0 + (*-1)\Delta t$ (for $*=1,\ 2,\ \ldots,\ M+1$).

Differentiating (5.3.19) with respect to t, we obtain:

$$\frac{d}{dt}[T^{(n)}(t)] \approx \sum_{l=1}^{M+1} \left(T^{(n)}(t^{(l)}) \frac{\sum_{m=1, m\neq l}^{M+1} \prod_{j=1,\, j\neq l,\, j\neq m}^{M+1} (t-t^{(j)})}{\prod_{\lambda=1, k\neq l}^{M+1} (t^{(l)}-t^{(k)})} \right)$$

(5.3.23)

Substituting (5.3.22)-(5.3.23) into the system of ordinary differential equations, and if t (in the system of ordinary differential equations) is chosen to be given in turn by $t^{(1)}, t^{(2)}, \ldots t^{(M)}$ and $t^{(M+1)}$, we obtain a system of linear algebraic equations containing $T^{(n)}(t^{(*)})$ for $(n=1,\ 2,\ \ldots,2N+L$ and $*=1,\ 2,\ \ldots,M+1)$ and $q^{(m)}(t^{(*)})$ for $(m=1, 2,\ldots,2N$ and $*=1, 2,\ldots,M+1)$, and hence the temperature is determined in the region R.

5.4 Example

For a specific problem governed by (5.2.1) with coefficients given by

$$\frac{1}{2}\lambda_{11} = \lambda_{12} = \lambda_{21} = \frac{2}{3}\lambda_{22} = 3+\cos^2\left(\frac{x}{2}+\frac{y}{3}\right),$$

$$\rho c = 3\left[1+\cos^2\left(\frac{x}{2}+\frac{y}{3}\right)\right].$$

(5.4.24)

We use a particular solution

$$T(x,y,t) = x - \frac{4}{3}y + 2 + e^{-t}\cos\left(\frac{x}{2}+\frac{y}{3}\right)$$

(5.4.25)

Thus, we can choose the region R to be given by

$$R = \{(x, y) : 0 < x < 1, 0 < y < 1\} \tag{5.4.26}$$

The dual-reciprocity boundary element method is applied to solve (5.2.1) with coefficients (5.4.24) and (5.4.25) inside the square domain subject to the initial-boundary conditions

$$T(x, y, 0) = f(x, y) \quad \text{for} \quad (x, y) \in R \cup D$$
$$T(x, y, t) = g(x, y, t) \quad \text{for} \quad (x,y) \in D_1 \text{ and } t > 0$$
$$q(x, y, t) = h(x, y, t) \quad \text{for} \quad (x, y) \in D_2 \quad \text{and} \quad t > 0 \tag{5.4.27}$$

Points	BEM N=60, t=1	BEM N=120, t=1	FDM, t=1	Exact, t=1
(0.1,0.2)	2.1998	2.1984	2.1982	2.1987
(0.1,0.3)	2.0646	2.0638	2.0635	2.0637
(0.1,0.4)	1.9295	1.9285	1.9283	1.9284
(0.5,0.2)	2.5838	2.5829	2.5819	2.5829
(0.5,0.3)	2.4464	2.4457	2.4398	2.4456
(0.5,0.4)	2.3086	2.3079	2.3079	2.3078
(0.9,0.2)	2.9545	2.9535	2.9486	2.9532
(0.9,0.3)	2.8146	2.8138	2.7937	2.8136
(0.9,0.4)	2.6745	2.6739	2.6589	2.6737

Table (5.1)

The result, we found that the boundary element solution agrees quite well with the exact solution. And more suitable than the result we obtained from the finite difference method.

Discussion of Results

As mentioned in the introduction of the thesis, exact closed form solutions are seldom available to most of the heat conduction and the use of approximate method is often inevitable. This thesis introduces the results of the most popular computational method available for solving heat conduction problems, the finite difference method and the boundary element method. Both these methods are based on the philosophy of discretisation and they provide approximate solutions to a large variety of heat conduction problems. The finite difference method is a domain-type method where the entire problem domain is discretized into finite elements. On the other hand, in the boundary element method the boundary of the region alone needs to be discretized. As mentioned in the introduction of this thesis many researchers solved the problems of temperature distribution analytically or with using finite difference method but we used the boundary element technique for solving the problems under consideration in our thesis for these reasons.

1) The boundary element method (BEM) has received much attention from researchers and has become an important technique in the computational solution of a method of physical problems. In common with the better-known finite difference method (FDM), the boundary element method is essentially a method for solving partial differential equations (PDEs) and can only be employed when the physical problem can be expressed as such as with the other method mentioned, the boundary element method is a numerical method and hence it is an important subject of research amongst the numerical analysis community. However, the potential advantages of the BEM have seemed so considerable that the strongest impetus behind its development has come from the engineering community, in its enthusiasm to obtain flexible and efficient computer-based solutions to a range of engineering problems.

2) The advantages in the boundary element method arises from the fact that only the boundary (or boundaries) of the domain of the PDE requires sub-division. (In the finite difference method the whole domain of the PDE requires discretisation). Thus the dimension of the problem is effectively reduced by one for example an equation governing a three-dimensional region is transformed into one over its surface. In cases where the domain is exterior to the boundary, as it is in potential flow past an obstacle, the extent of the domain is infinite an hence the advantages of the BEM are even more striking; the equation governing the infinite domain is reduced to an equation over the (finite) boundary.

3) The friendliness and openness of the BEM Community and its ability to continue to grow by attracting younger researchers all the time.
4) The major motivation behind the method of BEM was to reduce the dependency of the analysis on the definition of meshes. This motivation allowed the method to expand naturally, into new areas such as Dual Reciprocity, Complex Variable that will be used in this thesis and other Mesh Reduction Techniques.
5) The importance that BEM attached, right from the beginning, to produce industrial application tools.
6) We announce that the boundary element method solutions agree quite well with the exact solution, and more suitable than the result we obtained from the finite difference method or and the finite element method.

RESULTS AND RECOMMENDATION

[1] The present thesis concludes generally that the boundary element method is more suitable for numerical study of difficult thermal science problem than the finite difference method. Also, boundary element method will be an important technique in the computational solution of a number of physical, engineering and scientific problems.

[2] Boundary element method will be an important technique may be used in a variety of areas in engineering science, such as potential theory, thermo-elastostatics, thermo-elastodynamics, thermo-elastoplasticity, thermo-fluid mechanics, heat conduction in any anisotropic media.

[3] Boundary element method will be an important technique in thermo-electromagnetism and thermopiezoelectric.

[4] Complex variable boundary element method will be suitable for solving complex static problems.

[5] Dual reciprocity boundary element method will be suitable for solving complex dynamic problems.

[6] We recommende that it is very important and suitable for those doing in area of industrial mathematics and physics and engineering to use boundary element method in their research areas and works.

```
Warning: MATLAB Toolbox Path Cache is out of date and is not being used.
Type 'help toolbox_path_cache' for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> x=0.1, y= 0.2:0.1:0.4
T1=(sinh(pi.*x)).*(cos(pi.*y))
T2=sinh(pi)
T=T1/T2
x=0.5, y= 0.2:0.1:0.4
T1=(sinh(pi.*x)).*(cos(pi.*y))
T2=sinh(pi)
T=T1/T2
x=0.9, y= 0.2:0.1:0.4
T1=(sinh(pi.*x)).*(cos(pi.*y))
T2=sinh(pi)
T=T1/T2

x =
    0.1000
y =
    0.2000    0.3000    0.4000
T1 =
    0.2584    0.1877    0.0987
T2 =
   11.5487

T =
    0.0224    0.0163    0.0085


x =
    0.5000
y =
    0.2000    0.3000    0.4000
T1 =
    1.8618    1.3527    0.7111
T2 =
   11.5487

T =
    0.1612    0.1171    0.0616

x =
    0.9000
y =
    0.2000    0.3000    0.4000

T1 =
    6.8131    4.9500    2.6024
```

```
T2 =
  11.5487

T =
   0.5899    0.4286    0.2253

>>
Warning: MATLAB Toolbox Path Cache is out of date and is not being used.
Type 'help toolbox_path_cache' for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> x=0.1, y= 0.2:0.1:0.4, t=1
T1=exp((-(pi.^2).*t)/8)
T2=(cos((pi.*x)/4).*sin((pi.*y)/4))
T=1+(T1.*T2)
x=0.5, y= 0.2:0.1:0.4, t=1
T1=exp((-(pi.^2).*t)/8)
T2=(cos((pi.*x)/4).*sin((pi.*y)/4))
T=1+(T1.*T2)
x=0.9, y= 0.2:0.1:0.4, t=1
T1=exp((-(pi.^2).*t)/8)
T2=(cos((pi.*x)/4).*sin((pi.*y)/4))
T=1+(T1.*T2)

x =
   0.1000
y =
   0.2000    0.3000    0.4000
t =
     1
T1 =
   0.2912
T2 =
   0.1560    0.2327    0.3081

T =
   1.0454    1.0678    1.0897

x =
   0.5000
y =
   0.2000    0.3000    0.4000
t =
     1
T1 =
   0.2912
T2 =
   0.1445    0.2157    0.2855
```

```
T =
   1.0421    1.0628    1.0831

x =
   0.9000
y =
   0.2000    0.3000    0.4000
t =
     1
T1 =
   0.2912
T2 =
   0.1190    0.1775    0.2350

T =
   1.0346    1.0517    1.0684

>>Warning: MATLAB Toolbox Path Cache is out of date and is not being used.
Type 'help toolbox_path_cache' for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> x=0.1, y=0.2:0.1:0.4
T1=(4.*(x.^2))+(10.*(x.*y))
T2=(-7.*(y.^2))+(20.*x)+(18.*y)
T=T1+T2
x=0.5, y=0.2:0.1:0.4
T1=(4.*(x.^2))+(10.*(x.*y))
T2=(-7.*(y.^2))+(20.*x)+(18.*y)
T=T1+T2
x=0.9, y=0.2:0.1:0.4
T1=(4.*(x.^2))+(10.*(x.*y))
T2=(-7.*(y.^2))+(20.*x)+(18.*y)
T=T1+T2

x =
   0.1000
y =
   0.2000    0.3000    0.4000
T1 =
   0.2400    0.3400    0.4400
T2 =
   5.3200    6.7700    8.0800

T =
   5.5600    7.1100    8.5200

x =
   0.5000
```

```
y =
   0.2000    0.3000    0.4000
T1 =
   2.0000    2.5000    3.0000
T2 =
  13.3200   14.7700   16.0800

T =
  15.3200   17.2700   19.0800

x =
   0.9000
y =
   0.2000    0.3000    0.4000
T1 =
   5.0400    5.9400    6.8400
T2 =
  21.3200   22.7700   24.0800

T =
  26.3600   28.7100   30.9200

>>Warning: MATLAB Toolbox Path Cache is out of date and is not being used.
Type 'help toolbox_path_cache' for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> x=0.1, y= 0.2:0.1:0.4, t=1
T1=x-(1.3333.*y)+2
T2=exp(-t).*cos((0.5.*x)+(0.3333.*y))
T=T1+T2
x=0.5, y= 0.2:0.1:0.4, t=1
T1=x-(1.3333.*y)+2
T2=exp(-t).*cos((0.5.*x)+(0.3333.*y))
T=T1+T2
x=0.9, y= 0.2:0.1:0.4, t=1
T1=x-(1.3333.*y)+2
T2=exp(-t).*cos((0.5.*x)+(0.3333.*y))
T=T1+T2

x =
   0.1000
y =
   0.2000    0.3000    0.4000
t =
     1
T1 =
   1.8333    1.7000    1.5667
```

```
T2 =
   0.3654    0.3637    0.3617

T =
   2.1987    2.0638    1.9284

x =
   0.5000
y =
   0.2000    0.3000    0.4000
t =
     1
T1 =
   2.2333    2.1000    1.9667
T2 =
   0.3496    0.3456    0.3412

T =
   2.5829    2.4456    2.3079

x =
   0.9000
y =
   0.2000    0.3000    0.4000
t =
     1
T1 =
   2.6333    2.5000    2.3667
T2 =
   0.3199    0.3136    0.3070
T =
   2.9532    2.8136    2.6737

>>
```

```
      Program EX1PT1
      integer NO , BCT(1000) , N , i , ians
      double precision xb(1000) , yb(1000) , xm(1000) , ym(1000) ,
   &  nx(1000) , ny(1000) , lg(1000) , BCV(1000),
   &  phi(1000) , dphi(1000) , pint , d1 , xi , eta , pi

      print*,'Enter number of elements per side (<250):'
      red*,N0
      N=4*N0
      pi=4d0*dtan(1d0)
      d1=1d0/dfloat(N0)

      do 10 i=1,N0
      xb(i)=df1oat(i-1)*d1
      yb(i)=0d0
      xb(N0+i)=1d0
      yb(N0+i)xb(i)
      xb(2*N0+i)=1d0-xb(i)
      yb(2*N0+i)=1d0
      xb(3*N0+i)=0d0
      yb(3*N0+i)=1d0-xb(i)
   10 continue
      xb(N+1)=xb(1)
      yb(N+1)=yb(1)

      do 20 i=1 , N
      xm(i)=0.5do*(xb(i)+xb(i+1))
      ym(i)=0.5do*(yb(i)+yb(i+1))
      1g(i)=dsqrt((xb(i+1)-xb(i))**2d0+(yb(i+1)-yb(i))**2d0)
      nx(i)=(yb(i+1)-yb(i))/1g(i)
      ny(i)=(xb(i)-xb(i+1))/1g(i)
   20 continue

      do 30 i=1 ,N
      if (i.1e.N0) then
      BCT(i)=1
      BCV(i)=0d0
      else if  ((i.gt.N0).and.(i.le.(2*N0)))  then
      BCT(i)=0
      BCV(i)=dcos(pi*ym(i))
      else if  ((i.gt.(2*N0)).and.(i.le.(3*N0)))  then
      BCT(i)=1
      BCV(i)=0d0
      else
      BCT(i)=0
      BCV(i)=0d0
      endif
   30 continue
```

```
    call CELAP1 (N, xm, ym, xb, yb, nx, ny, 1g, BCT, BCV, phi, dphi)
50  print*,'Enter coordinates xi and eta of an interior point:'

    read*,xi,eta

    call CELAP2(N, xi, eta, xb, yb, nx, ny, 1g, phi, dphi, pint)

    write (*,60) pint , (dexp(pi*xi)-dexp(-pi*xi))*dcos(pi*eta)
 &  /(dexp(pi)-dexp(-pi))
60  format ('Numerical and exact values are:',
 &  F14.6,' and',F14.6,' respectively')

    print*,'To continue with another point enter 1:'
    read*, ians

    if (ians.eq.1) goto 50

    end
```

```
      program EX2PT1

      integer N0, BCT(1000), N, i , ians, N1, L, j, k,1ud

      double precision xb(1000), yb(1000), xm(1000), ym(1000), d1, alpha,
      & nx(1000), ny(1000), 1g(1000), BCV(1000), pi, tau , phi(1000), dt,
      & ti, tir

      print*,'Enter integer N0 (<101):'
      read*, N0
      N=4*N0

      print*,'Enter integer N1 (<15):'
      read *,N1
      L=N1**2
      NL=2*N+L
      print*,'Enter time-step dt:'
      read*,dt
      tau=0.25d0

      pi=4d0*datan(1d0)
      alpha=1d0

      d1=1d0/dfloat(N0)

      do 10 i=1,N0
      xb(i)=dfloat(i-1)*d1
      yb(i)=0d0
      xb(N0+i)=1d0
      yb(N0+i)=xb(i)
      xb(2*N0+i)=1d0-xb(i)
      yb(2*N0+i)=1d0
      xb(3*N0+i)=0d0
      yb(3*N0+i)=xb(2*N0+i)

10    continue

      xb(N+1)=xb(1)
```

```
      yb(N+1)=yb(1)

      do 20 i=1,N
      xm(i)=xb(i)+tau*(xb(i+1)-xb(i))
      ym(i)=yb(i)+tau*(yb(i+1)-yb(i))
      xm(N+i)=xb(i)+(1d0-tau)*(xb(i+1)-xb(i))
      ym(N+i)=yb(i)+(1d0-tau)*(yb(i+1)-yb(i))
      1g(i)=dsqrt((xb(i+1)-xb(i))**2d0+(yb(i+1)-yb(i))**2d0)
      nx(i)=(yb(i+1)-yb(i))/1g(i)
      nx(i)=(xb(i)-xb(i+1))/1g(i)
   20 continue

      d1=1d0/dfloat(N1+1)

      i=2*N

      do 25 j=1,N1
      do25 k=1,N1
      i=i+1
      xm(i)=dfloat(j)*d1
      ym(i)=dfloat(k)*d1
   25 continue

      do 26 i=1,N
      if ((i.le.N0).or.((i.gt(2*N0)).and.(i.le.(3*N0)))) then
      BCT(i)=1
      else
      BCT(i)=0
      endif
   26 continue

      do 27 i=1,NL
      phi(i)=1d0+dcos(0,25d0*pi*xm(i))*dsin(0.25d0*pi*ym(i))
   27 continue

      ti=-0,5d0*dt
      1du=1

   28 ti=ti+dt
      tir=ti+0.5d0*dt

      do 30 i=1,N
```

```
    if (i.le.N0) then
    BCV(i)=-0.25d0*pi*dexp(-pi*pi*0.125d0*ti)
    & *dcos(0.25d0*pi*xm(i))
    BCV(N=i)=-0.25d0*pi*dexp(-pi*pi*0.125d0*ti)
    & *dcos(0,25d0*pi*xm(N+i))
    else if (i.le.(2*N0)) then
    BCV(i)=1d0+dexp(-pi*pi*0.125d0*tir)
    & dsin(0,25d0*pi*ym(i))/dsqrt(2d0)
    BCV(N+i)=1d0+dexp(-pi*pi*0.125d0tir)
    & dsin(0,25d0*pi*ym(N+i))/dsqrt(2d0)
    else if (i.le(3*N0)) then
    BCV(i)=0.25d0*pi*dexp(-pi*pi*0.125d0ti)
    & dcos(0,25d0*pi*xm(i))/dsqrt(2d0)
    BCV(N+i)=0.25d0*pi*dexp(-pi*pi*0.125d0tir)
    & *dcos(0.25d0*pi*xm(N+i))/dsqrt(2d0)
    else
    BCV(i)=1d0+dexp(-pi*pi*0.125d0tir)
    & dsin(0,25d0*pi*ym(i))
    BCV(N+i)=1d0+dexp(-pi*pi*0.125d0tir)
    & dsin(0,25d0*pi*ym(N+i))
    endif
30  continue

    call DLEDIFF(1ud,N,L,tau,alpa,xm,ym,
    & xb,yb,nx,ny,lg,BCT,BCV,dt,phi)

    print*,’Time=’,tir
    print*,’ x y Numerical Exact’

    do 50 i=2*N+1,2*N+L
    write (*,60) xm(i) , ym(i) ,phi(i) ,
    & 1d0+dexp(-pi*pi*0.125d0*tir)
    & *dcos(0.25d0*pi*xm(i))*dsin(0.25d0*pi*ym(i))
50  continue

60  format (4F14.6)

    print*,’To continue with the next time level enter 1:’
    read*, ians
    1ud=0
    if (ians.eq.1) goto 28

    end
```

MAIN PROGRAM

VARIABLES

T:	TEMPERTAURE ARRAY
Q:	HEAT FLUX ARRAY
BC	BOUNDARY CONDITION ARRAY
HB	BOUNADRY FILM COEFFICIENT ARRAY
KB	BOUNDARY CONDITION CODE ARRAY
	(1) –FIRST KIND (TEMPERATURE
	(2) –SECOND KIND (HEST FLUX)
	(3) _THRD KIND (CONVECTION)
X,Y:	BOUNDARY ELEMENT NODAL COORDINATES ARRAY
XI,YI:	INTERNAL POINT COORDINATE ARRAY
E,F:	INFLUENCE COEFFICIENT MATRICES
C:	THERMAL CONDUCTIVITY COEFFICIENT ARRAY
NE:	NUMBER OF BOUNDARY ELEMENTS
NI:	NUMBER OF INTERNAL POINTS
NS:	NUMBER OF SAMBLE POINTS FOR K
X1,Y1:	
X2,Y2:	
X3,Y3:	CURRENT ELEMENT NODAL POINT COORDINATES
XP,YP:	CURRENT COLLOCAYION POINT

```
     IMPLICIT DOBLE PRECISION (A-H,O-Z)

     INCLUDE 'Bem2D parameters. for'

     REAL*8 T(NEMAX+NIMAX),Q(NEMAX)
     REAL*8 BC(NEMAX),HB(NEMAX)
     REAL*8 X(NEMAX,3),Y(NEMAX,3)
     REAL*8 XI(NEMAX),YI(NEMAX)
     REAL*8 E(NEMAX+NIMAX, NEMAX+NIMAX)
     REAL*8 F(NEMAX+NIMAX, NEMAX+NIMAX)
     REAL*8 BM(NEMAX+NIMAX)
     REAL*8 W(NEMAX+NIMAX)
     REAL*8 XS(NSMAX),YS(NSMAX),KS(NSMAX,3)
     REAL*8 KXMAX,KYMAX
     INTEGER KB(NEMAX)
     INTEGER NS
     INTEGER KCONST,KORTHO
     INTEGER INDX (NEMAX+NIMAX)
     CHARACTER*40 FILEIN
     CHARACTER*80 TITLE

     COMMON/CONDUC/(6,3)
     COMMON/INFORM/NE,NI
     COMMON/COEFFI/X1,X2,X3,Y1,Y2,Y3,XP,YP,PI

     INPUT DATA FILE AND INITIAL PARAMETERS

     WRITE(*,*)
     WRITE(*,*) 'WAIT WHILE BEM DATA IS BEING PROCESSED…'
     WRITE(*,*)

     WRITE (*,101)
     READ (*,'(A)') FILEIN
     OPEN (15,FILE=FILEIN,STATUS='OLD')
101  FORMAT(1X,'ENTERTHE PATH AND NAME OF THE BEM I/O FILE:..')
     WRITE (*,001)
001FORMAT(//2X, 'READING PROBLEM INFORMATION FROM FILE…………')
     READ (15,*) NB
     READ (15,*) NB
     READ (15, '(A)') TITEL
     READ (15,*) NE,NI,NS
```

```
      READ (15,*) NB

      CALL CONDUCTIVITY DISTRIBUTION SUBROTIN

      WRITE (*,003)
003 FORMAT (2X, 'EXPANDING CONDUCTIVITY SAMPLES INTO A FUNCTION........ ' )
      CALL CONDUCTIVITY (XS,YS, KS, NS, KCONST, KORTHO)

      CALL DATA INPUT SUBROUTINE

      WRITE(*,004)
004 FORMAT(2X, 'READING PROBLEM BEM DATA FROM FILE…..............................…..' )
      CALL INPUT (X, Y,XI, YI, BC, KB, HB, W)

      CALL DATA TRANSFORM SUBROUTINE

      IF (KORTHO. EQ.1) THEN
      WRITE(*,006)
    006 FORMAT(2X, 'TRANSFORMING BEM DATAACCORDING TO ORTHOTROPY...' )
      CALL TRANSFORMING (X, Y,XI, YI, BC, KB, HB, KXMAX, KYMAX)
      END IF

      CALL COMPUTATION OF EPSILON COEFFICIENTS SUBROUTINE

      IF (KCONST.EQ.0) THEN

      WRITE(*,007)
007 FORMAT (2X, 'COMPUTING SIFTING ERROR CORRECTION …...........................…' )
      CALL EPSILONCOEFF (X, Y, XI, YI, W, E, F, BM, INDX)
      END IF

      CALL INFLUENCE COEFFICIENTS COMPUTATION SUBROUTINE

      WRITE(*,008)
008 FORMAT (2X, 'GENERATING INFLUENCE COEFFICIENT MATRICES…................' )
      CALL COMPUTING (X,Y, XI,YI, E, F,W)

      CALL ALGEBRAIC SETUP SUBROUTINE

      WRITE(*,010)
010 FORMAT (2X, 'ARRANGING AND SOLVING ALGEBRAIC SYSTEM…................…' )
      CALL ALGEBRA (E, F, BC, KB, HB, BM, INDX, T, Q)

      CALL DATA BACK-TRANSFORM SUBROUTINE

      IF (KORTHO.EQ,1) THEN
      WRITE(*,011)
011 FORMAT (2X, 'BACKTRANSFORMING DATA ACCORDING TO ORTHOTROPY... ' )
      CALL BACKTRANSFORM (X, Y, XI, YI, Q, KB, BC, HB, KXMAX, KYMAX)
      END IF

    CALL DATA OUTPUT SUBROUTINE

    WRITE(*,013)
013 FORMAT (2X, 'OUTPUTTING BEM RESULTS TO BEM I/O FILE …......................…' )
    CALL OUTPUT (T,Q)

    CLOSE (15)

    WRITE (*,*)
    WRITE (*,*) '…BEM ANALYSIS COMPLETED SUCCESSFULY'
    WRITE (*,*)
    WRITE (*,*) 'PRESS <ENTER> TO END PROGRAM'
    READ (*,*)

    END
```

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